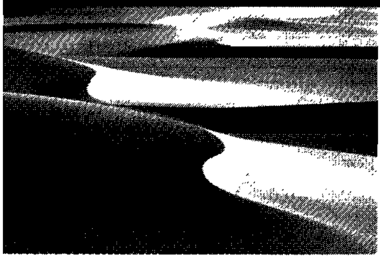
CHAPTER P



# Preliminaries

> ‘Reeling and Writhing, of course, to begin with,’ the Mock Turtle replied, ‘and the different branches of Arithmetic — Ambition, Distraction, Uglification, and Derision.’

Lewis Carroll (Charles Lutwidge Dodgson) 1832–1898
from *Alice's Adventures in Wonderland*

## Introduction

This preliminary chapter reviews the most important things you should know before beginning calculus. Topics include the real number system, Cartesian coordinates in the plane, equations representing straight lines, circles, and parabolas, functions and their graphs, and, in particular, polynomials and trigonometric functions.

Depending on your precalculus background, you may or may not be familiar with these topics. If you are, you may want to skim over this material to refresh your understanding of the terms used; if not, you should study this chapter in detail.

## P.1 Real Numbers and the Real Line

Calculus depends on properties of the real number system. **Real numbers** are numbers that can be expressed as decimals, for example,

$$\begin{aligned} 5 &= 5.00000\ldots \\ -\tfrac{3}{4} &= -0.750000\ldots \\ \tfrac{1}{3} &= 0.3333\ldots \\ \sqrt{2} &= 1.4142\ldots \\ \pi &= 3.14159\ldots \end{aligned}$$

In each case the three dots ... indicate that the sequence of decimal digits goes on forever. For the first three numbers above, the patterns of the digits are obvious; we know what all the subsequent digits are. For $\sqrt{2}$ and $\pi$ there are no obvious patterns.

The real numbers can be represented geometrically as points on a number line, which we call the **real line**, shown in Figure P.1. The symbol $\mathbb{R}$ is used to denote either the real number system or, equivalently, the real line.

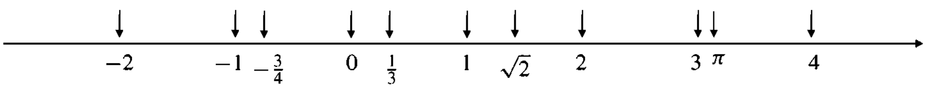



**Figure P.1** The real line

The properties of the real number system fall into three categories: algebraic properties, order properties, and completeness. You are already familiar with the *algebraic properties*; roughly speaking, they assert that real numbers can be added, subtracted, multiplied, and divided (except by zero) to produce more real numbers and that the usual rules of arithmetic are valid.

The *order properties* of the real numbers refer to the order in which the numbers appear on the real line. If $x$ lies to the left of $y$, then we say that "$x$ is less than $y$" or "$y$ is greater than $x$." These statements are written symbolically as $x < y$ and $y > x$, respectively. The inequality $x \le y$ means that either $x < y$ or $x = y$. The order properties of the real numbers are summarized in the following *rules for inequalities*:

The symbol $\Longrightarrow$ means "implies."

**Rules for inequalities**

If $a$, $b$, and $c$ are real numbers, then:

1. $a < b \quad \Longrightarrow \quad a + c < b + c$
2. $a < b \quad \Longrightarrow \quad a - c < b - c$
3. $a < b$ and $c > 0 \quad \Longrightarrow \quad ac < bc$
4. $a < b$ and $c < 0 \quad \Longrightarrow \quad ac > bc$; in particular, $-a > -b$
5. $a > 0 \quad \Longrightarrow \quad \dfrac{1}{a} > 0$
6. $0 < a < b \quad \Longrightarrow \quad \dfrac{1}{b} < \dfrac{1}{a}$

Rules 1–4 and 6 (for $a > 0$) also hold if $<$ and $>$ are replaced by $\le$ and $\ge$.

Note especially the rules for multiplying (or dividing) an inequality by a number. If the number is positive, the inequality is preserved; if the number is negative, the inequality is reversed.

The *completeness* property of the real number system is more subtle and difficult to understand. One way to state it is as follows: if $A$ is any set of real numbers having at least one number in it, and if there exists a real number $y$ with the property that $x \le y$ for every $x$ in $A$ (such a number $y$ is called an **upper bound** for $A$), then there exists a *smallest* such number, called the **least upper bound** or **supremum** of $A$, and denoted $\sup(A)$. Roughly speaking, this says that there can be no holes or gaps on the real line—every point corresponds to a real number. We will not need to deal much with completeness in our study of calculus. It is typically used to prove certain important results, in particular, Theorems 8 and 9 in Chapter 1. (These proofs are given in Appendix III but are not usually included in elementary calculus courses; they are studied in more advanced courses in mathematical analysis.) However, when we study infinite sequences and series in Chapter 9, we will make direct use of completeness.

The set of real numbers has some important special subsets:

(i) the **natural numbers** or **positive integers**, namely, the numbers $1, 2, 3, 4, \ldots$

(ii) the **integers**, namely, the numbers $0, \pm 1, \pm 2, \pm 3, \ldots$

(iii) the **rational numbers**, that is, numbers that can be expressed in the form of a fraction $m/n$, where $m$ and $n$ are integers, and $n \ne 0$.

The rational numbers are precisely those real numbers with decimal expansions that are either:

(a) terminating, that is, ending with an infinite string of zeros, for example, $3/4 = 0.750000\ldots$, or

(b) repeating, that is, ending with a string of digits that repeats over and over, for example, $23/11 = 2.090909\ldots = 2.\overline{09}$. (The bar indicates the pattern of repeating digits.)

Real numbers that are not rational are called *irrational numbers*.

**EXAMPLE 1** Show that each of the numbers (a) $1.323232\cdots = 1.\overline{32}$ and (b) $0.3405405405\ldots = 0.3\overline{405}$ is a rational number by expressing it as a quotient of two integers.

***Solution***

(a) Let $x = 1.323232\ldots$ Then $x - 1 = 0.323232\ldots$ and

$$100x = 132.323232\ldots = 132 + 0.323232\ldots = 132 + x - 1.$$

Therefore, $99x = 131$ and $x = 131/99$.

(b) Let $y = 0.3405405405\ldots$ Then $10y = 3.405405405\ldots$ and $10y - 3 = 0.405405405\ldots$ Also,

$$10000y = 3405.405405405\ldots = 3405 + 10y - 3.$$

Therefore, $9990y = 3402$ and $y = 3402/9990 = 63/185$.

The set of rational numbers possesses all the algebraic and order properties of the real numbers but not the completeness property. There is, for example, no rational number whose square is 2. Hence, there is a "hole" on the "rational line" where $\sqrt{2}$ should be.[1] Because the real line has no such "holes," it is the appropriate setting for studying limits and therefore calculus.

## Intervals

A subset of the real line is called an **interval** if it contains at least two numbers and also contains all real numbers between any two of its elements. For example, the set of real numbers $x$ such that $x > 6$ is an interval, but the set of real numbers $y$ such that $y \neq 0$ is not an interval. (Why?) It consists of two intervals.

If $a$ and $b$ are real numbers and $a < b$, we often refer to

(i) the **open interval** from $a$ to $b$, denoted by $(a, b)$, consisting of all real numbers $x$ satisfying $a < x < b$.

(ii) the **closed interval** from $a$ to $b$, denoted by $[a, b]$, consisting of all real numbers $x$ satisfying $a \le x \le b$.

(iii) the **half-open interval** $[a, b)$, consisting of all real numbers $x$ satisfying the inequalities $a \le x < b$.

(iv) the **half-open interval** $(a, b]$, consisting of all real numbers $x$ satisfying the inequalities $a < x \le b$.

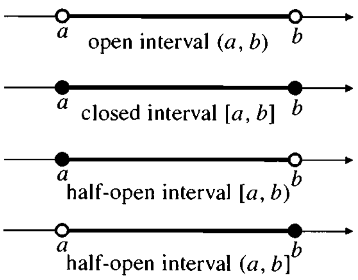



**Figure P.2** Finite intervals

These are illustrated in Figure P.2. Note the use of hollow dots to indicate endpoints of intervals that are not included in the intervals, and solid dots to indicate endpoints that are included. The endpoints of an interval are also called **boundary points**.

The intervals in Figure P.2 are **finite intervals**; each of them has finite length $b-a$. Intervals can also have infinite length, in which case they are called **infinite intervals**. Figure P.3 shows some examples of infinite intervals. Note that the whole real line $\mathbb{R}$ is an interval, denoted by $(-\infty, \infty)$. The symbol $\infty$ ("infinity") does *not* denote a real number, so we never allow $\infty$ to belong to an interval.

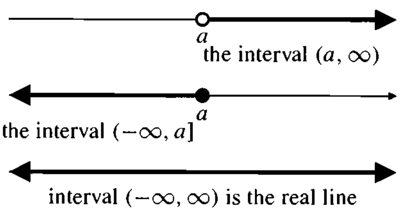



**Figure P.3** Infinite intervals

[1] How do we know that $\sqrt{2}$ is an irrational number? Suppose, to the contrary, that $\sqrt{2}$ is rational. Then $\sqrt{2} = m/n$, where $m$ and $n$ are integers and $n \neq 0$. We can assume that the fraction $m/n$ has been "reduced to lowest terms"; any common factors have been cancelled out. Now $m^2/n^2 = 2$, so $m^2 = 2n^2$, which is an even integer. Hence $m$ must also be even. (The square of an odd integer is always odd.) Since $m$ is even, we can write $m = 2k$, where $k$ is an integer. Thus $4k^2 = 2n^2$ and $n^2 = 2k^2$, which is even. Thus $n$ is also even. This contradicts the assumption that $\sqrt{2}$ could be written as a fraction $m/n$ in lowest terms; $m$ and $n$ cannot both be even. Accordingly, there can be no rational number whose square is 2.

**EXAMPLE 2** Solve the following inequalities. Express the solution sets in terms of intervals and graph them.

(a) $2x - 1 > x + 3$ (b) $-\dfrac{x}{3} \ge 2x - 1$ (c) $\dfrac{2}{x-1} \ge 5$

### *Solution*

(a) $2x - 1 > x + 3$ Add 1 to both sides.

$2x > x + 4$ Subtract $x$ from both sides.

$x > 4$ The solution set is the interval $(4, \infty)$.

(b) $-\dfrac{x}{3} \ge 2x - 1$ Multiply both sides by $-3$.

$x \le -6x + 3$ Add $6x$ to both sides.

$7x \le 3$ Divide both sides by 7.

$x \le \dfrac{3}{7}$ The solution set is the interval $(-\infty, 3/7]$.

(c) We transpose the 5 to the left side and simplify to rewrite the given inequality in an equivalent form:

$$\frac{2}{x-1} - 5 \ge 0 \quad \Longleftrightarrow \quad \frac{2 - 5(x-1)}{x-1} \ge 0 \quad \Longleftrightarrow \quad \frac{7 - 5x}{x-1} \ge 0.$$

The symbol $\Longleftrightarrow$ means "if and only if" or "is equivalent to." If $A$ and $B$ are two statements, then $A \Longleftrightarrow B$ means that the truth of either statement implies the truth of the other, so either both must be true or both must be false.

The fraction $\dfrac{7-5x}{x-1}$ is undefined at $x = 1$ and is 0 at $x = 7/5$. Between these numbers it is positive if the numerator and denominator have the same sign, and negative if they have opposite sign. It is easiest to organize this sign information in a chart:

| $x$ | | 1 | | 7/5 | |
|---|---|---|---|---|---|
| $7 - 5x$ | + | + | + | 0 | − |
| $x - 1$ | − | 0 | + | + | + |
| $(7 - 5x)/(x - 1)$ | − | undef | + | 0 | − |

Thus the solution set of the given inequality is the interval $(1, 7/5]$.

See Figure P.4 for graphs of the solutions.

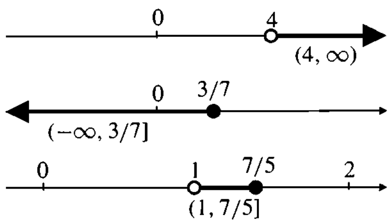



**Figure P.4** The intervals for Example 2

Sometimes we will need to solve systems of two or more inequalities that must be satisfied simultaneously. We still solve the inequalities individually and look for numbers in the intersection of the solution sets.

**EXAMPLE 3** Solve the systems of inequalities:

(a) $3 \le 2x + 1 \le 5$ (b) $3x - 1 < 5x + 3 \le 2x + 15$.

### *Solution*

(a) Using the technique of Example 2, we can solve the inequality $3 \le 2x + 1$ to get $2 \le 2x$, so $x \ge 1$. Similarly, the inequality $2x + 1 \le 5$ leads to $2x \le 4$, so $x \le 2$. The solution set of system (a) is therefore the closed interval $[1, 2]$.

(b) We solve both inequalities as follows:

$$\left.\begin{aligned} 3x - 1 &< 5x + 3 \\ -1 - 3 &< 5x - 3x \\ -4 &< 2x \\ -2 &< x \end{aligned}\right\} \quad \text{and} \quad \left\{\begin{aligned} 5x + 3 &\le 2x + 15 \\ 5x - 2x &\le 15 - 3 \\ 3x &\le 12 \\ x &\le 4 \end{aligned}\right.$$

The solution set is the interval $(-2, 4]$.

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Solving quadratic inequalities depends on solving the corresponding quadratic equations.

**EXAMPLE 4** **Quadratic inequalities**

Solve: (a) $x^2 - 5x + 6 < 0$ (b) $2x^2 + 1 > 4x$.

### *Solution*

(a) The trinomial $x^2 - 5x + 6$ factors into the product $(x-2)(x-3)$, which is negative if and only if exactly one of the factors is negative. Since $x - 3 < x - 2$, this happens when $x - 3 < 0$ and $x - 2 > 0$. Thus we need $x < 3$ and $x > 2$; the solution set is the open interval $(2, 3)$.

(b) The inequality $2x^2 + 1 > 4x$ is equivalent to $2x^2 - 4x + 1 > 0$. The corresponding quadratic equation $2x^2 - 4x + 1 = 0$, which is of the form $Ax^2 + Bx + C = 0$, can be solved by the quadratic formula (see Section P.6):

$$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} = \frac{4 \pm \sqrt{16 - 8}}{4} = 1 \pm \frac{\sqrt{2}}{2},$$

so the given inequality can be expressed in the form

$$\left(x - 1 + \tfrac{1}{2}\sqrt{2}\right)\left(x - 1 - \tfrac{1}{2}\sqrt{2}\right) > 0.$$

This is satisfied if both factors on the left side are positive or if both are negative. Therefore, we require that either $x < 1 - \frac{1}{2}\sqrt{2}$ or $x > 1 + \frac{1}{2}\sqrt{2}$. The solution set is the *union* of intervals $\left(-\infty, 1 - \frac{1}{2}\sqrt{2}\right) \cup \left(1 + \frac{1}{2}\sqrt{2}, \infty\right)$.

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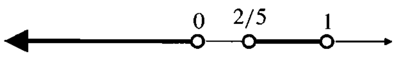
Note the use of the symbol $\cup$ to denote the **union** of intervals. A real number is in the union of intervals if it is in at least one of the intervals. We will also need to consider the **intersection** of intervals from time to time. A real number belongs to the intersection of intervals if it belongs to *every one* of the intervals. We will use $\cap$ to denote intersection. For example,

$$[1, 3) \cap [2, 4] = [2, 3) \quad \text{while} \quad [1, 3) \cup [2, 4] = [1, 4].$$

**EXAMPLE 5** Solve the inequality $\dfrac{3}{x-1} < -\dfrac{2}{x}$ and graph the solution set.

***Solution*** We would like to multiply by $x(x-1)$ to clear the inequality of fractions, but this would require considering three cases separately. (What are they?) Instead, we will transpose and combine the two fractions into a single one:




the union $(-\infty, 0) \cup (2/5, 1)$

**Figure P.5** The solution set for Example 5

$$\frac{3}{x-1} < -\frac{2}{x} \iff \frac{3}{x-1} + \frac{2}{x} < 0 \iff \frac{5x-2}{x(x-1)} < 0.$$

We examine the signs of the three factors in the left fraction to determine where that fraction is negative:

| $x$ | | 0 | | 2/5 | | 1 | |
|---|---|---|---|---|---|---|---|
| $5x-2$ | − | − | − | 0 | + | + | + |
| $x$ | − | 0 | + | + | + | + | + |
| $x-1$ | − | − | − | − | − | 0 | + |
| $\dfrac{5x-2}{x(x-1)}$ | − | undef | + | 0 | − | undef | + |

The solution set of the given inequality is the union of these two intervals, namely, $(-\infty, 0) \cup (2/5, 1)$. See Figure P.5.

## The Absolute Value

The **absolute value**, or **magnitude**, of a number $x$, denoted $|x|$ (read "the absolute value of $x$"), is defined by the formula

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

The vertical lines in the symbol $|x|$ are called **absolute value bars**.

**EXAMPLE 6** $|3| = 3$, $|0| = 0$, $|-5| = 5$.

Note that $|x| \geq 0$ for every real number $x$, and $|x| = 0$ only if $x = 0$. People sometimes find it confusing to say that $|x| = -x$ when $x$ is negative, but this is correct since $-x$ is positive in that case. The symbol $\sqrt{a}$ always denotes the *nonnegative* square root of $a$, so an alternative definition of $|x|$ is $|x| = \sqrt{x^2}$.

It is important to remember that $\sqrt{a^2} = |a|$. Do not write $\sqrt{a^2} = a$ unless you already know that $a \geq 0$.

Geometrically, $|x|$ represents the (nonnegative) distance from $x$ to 0 on the real line. More generally, $|x - y|$ represents the (nonnegative) distance between the points $x$ and $y$ on the real line, since this distance is the same as that from the point $x - y$ to 0 (see Figure P.6):

$$|x - y| = \begin{cases} x - y, & \text{if } x \geq y \\ y - x, & \text{if } x < y. \end{cases}$$

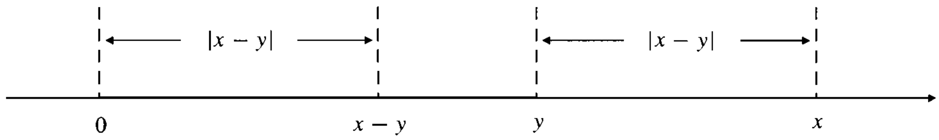



**Figure P.6**
$|x - y|$ = distance from $x$ to $y$

The absolute value function has the following properties:

**Properties of absolute values**

1. $|-a| = |a|$. A number and its negative have the same absolute value.
2. $|ab| = |a||b|$ and $\left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|}$. The absolute value of a product (or quotient) of two numbers is the product (or quotient) of their absolute values.
3. $|a \pm b| \leq |a| + |b|$ (the **triangle inequality**). The absolute value of a sum of or difference between numbers is less than or equal to the sum of their absolute values.

The first two of these properties can be checked by considering the cases where either of $a$ or $b$ is either positive or negative. The third property follows from the first two because $\pm 2ab \leq |2ab| = 2|a||b|$. Therefore, we have

$$\begin{aligned} |a \pm b|^2 &= (a \pm b)^2 = a^2 \pm 2ab + b^2 \\ &\leq |a|^2 + 2|a||b| + |b|^2 = (|a| + |b|)^2, \end{aligned}$$

and taking the (positive) square roots of both sides we obtain $|a \pm b| \leq |a| + |b|$. This result is called the "triangle inequality" because it follows from the geometric fact that the length of any side of a triangle cannot exceed the sum of the lengths of the other two sides. For instance, if we regard the points 0, $a$, and $b$ on the number line as the vertices of a degenerate "triangle," then the sides of the triangle have lengths $|a|$, $|b|$, and $|a - b|$. The triangle is degenerate since all three of its vertices lie on a straight line.

## Equations and Inequalities Involving Absolute Values

The equation $|x| = D$ (where $D > 0$) has two solutions, $x = D$ and $x = -D$: the two points on the real line that lie at distance $D$ from the origin. Equations and inequalities involving absolute values can be solved algebraically by breaking them into cases according to the definition of absolute value, but often they can also be solved geometrically by interpreting absolute values as distances. For example, the inequality $|x - a| < D$ says that the distance from $x$ to $a$ is less than $D$, so $x$ must lie between $a - D$ and $a + D$. (Or, equivalently, $a$ must lie between $x - D$ and $x + D$.) If $D$ is a positive number, then

$$\begin{aligned}
|x| = D \quad &\iff \quad \text{either } x = -D \text{ or } x = D \\
|x| < D \quad &\iff \quad -D < x < D \\
|x| \le D \quad &\iff \quad -D \le x \le D \\
|x| > D \quad &\iff \quad \text{either } x < -D \text{ or } x > D
\end{aligned}$$

More generally,

$$\begin{aligned}
|x - a| = D \quad &\iff \quad \text{either } x = a - D \text{ or } x = a + D \\
|x - a| < D \quad &\iff \quad a - D < x < a + D \\
|x - a| \le D \quad &\iff \quad a - D \le x \le a + D \\
|x - a| > D \quad &\iff \quad \text{either } x < a - D \text{ or } x > a + D
\end{aligned}$$

**EXAMPLE 7** Solve: (a) $|2x + 5| = 3$ (b) $|3x - 2| \le 1$.

### *Solution*

(a) $|2x + 5| = 3 \iff 2x + 5 = \pm 3$. Thus, either $2x = -3 - 5 = -8$ or $2x = 3 - 5 = -2$. The solutions are $x = -4$ and $x = -1$.

(b) $|3x - 2| \le 1 \iff -1 \le 3x - 2 \le 1$. We solve this pair of inequalities:

$$\left\{\begin{array}{c} -1 \le 3x - 2 \\ -1 + 2 \le 3x \\ 1/3 \le x \end{array}\right\} \quad \text{and} \quad \left\{\begin{array}{c} 3x - 2 \le 1 \\ 3x \le 1 + 2 \\ x \le 1 \end{array}\right\}.$$

Thus the solutions lie in the interval $[1/3, 1]$.

***Remark*** Here is how part (b) of Example 7 could have been solved geometrically, by interpreting the absolute value as a distance:

$$|3x - 2| = \left|3\left(x - \frac{2}{3}\right)\right| = 3\left|x - \frac{2}{3}\right|.$$

Thus the given inequality says that

$$3\left|x - \frac{2}{3}\right| \le 1 \quad \text{or} \quad \left|x - \frac{2}{3}\right| \le \frac{1}{3}.$$

This says that the distance from $x$ to $2/3$ does not exceed $1/3$. The solutions $x$ therefore lie between $1/3$ and 1, including both of these endpoints. (See Figure P.7.)

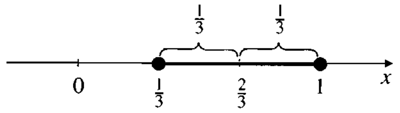



**Figure P.7** The solution set for Example 7(b)

**EXAMPLE 8** Solve the equation $|x+1| = |x-3|$.

***Solution*** The equation says that $x$ is equidistant from $-1$ and 3. Therefore, $x$ is the point halfway between $-1$ and 3; $x = (-1+3)/2 = 1$. Alternatively, the given equation says that either $x+1 = x-3$ or $x+1 = -(x-3)$. The first of these equations has no solutions; the second has the solution $x = 1$.

**EXAMPLE 9** What values of $x$ satisfy the inequality $\left|5 - \dfrac{2}{x}\right| < 3$?

***Solution*** We have

$$\begin{aligned}
\left|5 - \frac{2}{x}\right| < 3 \quad &\Longleftrightarrow \quad -3 < 5 - \frac{2}{x} < 3 && \text{Subtract 5 from each member.}\\
& \qquad -8 < -\frac{2}{x} < -2 && \text{Divide each member by } -2.\\
& \qquad 4 > \frac{1}{x} > 1 && \text{Take reciprocals.}\\
& \qquad \frac{1}{4} < x < 1.
\end{aligned}$$

In this calculation we manipulated a system of two inequalities simultaneously, rather than split it up into separate inequalities as we have done in previous examples. Note how the various rules for inequalities were used here. Multiplying an inequality by a negative number reverses the inequality. So does taking reciprocals of an inequality in which both sides are positive. The given inequality holds for all $x$ in the open interval $(1/4, 1)$.

## EXERCISES P.1

In Exercises 1–2, express the given rational number as a repeating decimal. Use a bar to indicate the repeating digits.

**1.** $\dfrac{2}{9}$ **2.** $\dfrac{1}{11}$

In Exercises 3–4, express the given repeating decimal as a quotient of integers in lowest terms.

**3.** $0.\overline{12}$ **4.** $3.2\overline{7}$

**5.** Express the rational numbers $1/7$, $2/7$, $3/7$, and $4/7$ as repeating decimals. (Use a calculator to give as many decimal digits as possible.) Do you see a pattern? Guess the decimal expansions of $5/7$ and $6/7$ and check your guesses.

**6.** Can two different decimals represent the same number? What number is represented by $0.999\ldots = 0.\overline{9}$?

In Exercises 7–12, express the set of all real numbers $x$ satisfying the given conditions as an interval or a union of intervals.

**7.** $x \ge 0$ and $x \le 5$ **8.** $x < 2$ and $x \ge -3$

**9.** $x > -5$ or $x < -6$ **10.** $x \le -1$

**11.** $x > -2$ **12.** $x < 4$ or $x \ge 2$

In Exercises 13–26, solve the given inequality, giving the solution set as an interval or union of intervals.

**13.** $-2x > 4$ **14.** $3x + 5 \le 8$

**15.** $5x - 3 \le 7 - 3x$ **16.** $\dfrac{6-x}{4} \ge \dfrac{3x-4}{2}$

**17.** $3(2-x) < 2(3+x)$ **18.** $x^2 < 9$

**19.** $\dfrac{1}{2-x} < 3$ **20.** $\dfrac{x+1}{x} \ge 2$

**21.** $x^2 - 2x \le 0$ **22.** $6x^2 - 5x \le -1$

**23.** $x^3 > 4x$ **24.** $x^2 - x \le 2$

**25.** $\dfrac{x}{2} \ge 1 + \dfrac{4}{x}$ **26.** $\dfrac{3}{x-1} < \dfrac{2}{x+1}$

Solve the equations in Exercises 27–32.

**27.** $|x| = 3$ **28.** $|x - 3| = 7$

**29.** $|2t + 5| = 4$ **30.** $|1 - t| = 1$

**31.** $|8 - 3s| = 9$ **32.** $\left|\dfrac{s}{2} - 1\right| = 1$

In Exercises 33–40, write the interval defined by the given inequality.

**33.** $|x| < 2$ **34.** $|x| \le 2$

**35.** $|s - 1| \le 2$

**36.** $|t + 2| < 1$

**37.** $|3x - 7| < 2$

**38.** $|2x + 5| < 1$

**39.** $\left|\frac{x}{2} - 1\right| \le 1$

**40.** $\left|2 - \frac{x}{2}\right| < \frac{1}{2}$

In Exercises 41–42, solve the given inequality by interpreting it as a statement about distances on the real line.

**41.** $|x + 1| > |x - 3|$

**42.** $|x - 3| < 2|x|$

**43.** Do not fall into the trap $|-a| = a$. For what real numbers $a$ is this equation true? For what numbers is it false?

**44.** Solve the equation $|x - 1| = 1 - x$.

**45.** Show that the inequality

$$|a - b| \ge \big||a| - |b|\big|$$

holds for all real numbers $a$ and $b$.

# P.2 Cartesian Coordinates in the Plane

The positions of all points in a plane can be measured with respect to two perpendicular real lines in the plane intersecting at the 0-point of each. These lines are called **coordinate axes** in the plane. Usually (but not always) we call one of these axes the $x$-axis and draw it horizontally with numbers $x$ on it increasing to the right; then we call the other the $y$-axis, and draw it vertically with numbers $y$ on it increasing upward. The point of intersection of the coordinate axes (the point where $x$ and $y$ are both zero) is called the **origin** and is often denoted by the letter $O$.

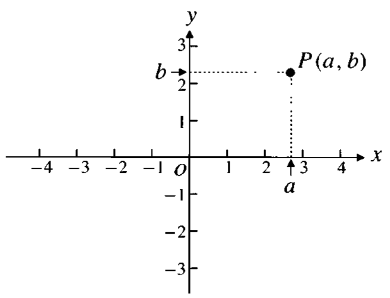



**Figure P.8** The coordinate axes and the point $P$ with coordinates $(a, b)$

If $P$ is any point in the plane, we can draw a line through $P$ perpendicular to the $x$-axis. If $a$ is the value of $x$ where that line intersects the $x$-axis, we call $a$ the **$x$-coordinate** of $P$. Similarly, the **$y$-coordinate** of $P$ is the value of $y$ where a line through $P$ perpendicular to the $y$-axis meets the $y$-axis. The **ordered pair** $(a, b)$ is called the **coordinate pair**, or the **Cartesian coordinates**, of the point $P$. We refer to the point as $P(a, b)$ to indicate both the name $P$ of the point and its coordinates $(a, b)$. (See Figure P.8.) Note that the $x$-coordinate appears first in a coordinate pair. Coordinate pairs are in one-to-one correspondence with points in the plane; each point has a unique coordinate pair, and each coordinate pair determines a unique point. We call such a set of coordinate axes and the coordinate pairs they determine a **Cartesian coordinate system** in the plane, after the seventeenth-century philosopher René Descartes, who created analytic (coordinate) geometry. When equipped with such a coordinate system, a plane is called a **Cartesian plane**. Note that we are using the same notation $(a, b)$ for the Cartesian coordinates of a point in the plane as we use for an open interval on the real line. However, this should not cause any confusion because the intended meaning will be clear from the context.

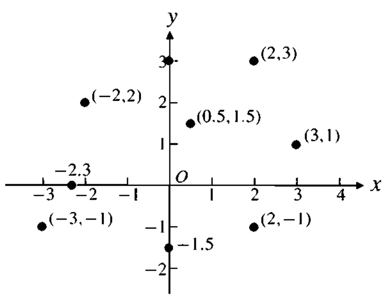



**Figure P.9** Some points with their coordinates

Figure P.9 shows the coordinates of some points in the plane. Note that all points on the $x$-axis have $y$-coordinate 0. We usually just write the $x$-coordinates to label such points. Similarly, points on the $y$-axis have $x = 0$, and we can label such points using their $y$-coordinates only.

The coordinate axes divide the plane into four regions called **quadrants**. These quadrants are numbered I to IV, as shown in Figure P.10. The **first quadrant** is the upper right one; both coordinates of any point in that quadrant are positive numbers. Both coordinates are negative in quadrant III; only $y$ is positive in quadrant II; only $x$ is positive in quadrant IV.

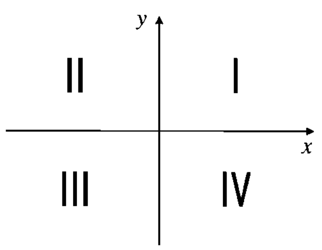



**Figure P.10** The four quadrants

## Axis Scales

When we plot data in the coordinate plane or graph formulas whose variables have different units of measure, we do not need to use the same scale on the two axes. If, for example, we plot height versus time for a falling rock, there is no reason to place the mark that shows 1 m on the height axis the same distance from the origin as the mark that shows 1 s on the time axis.

When we graph functions whose variables do not represent physical measurements and when we draw figures in the coordinate plane to study their geometry or trigonometry, we usually make the scales identical. A vertical unit of distance then looks the same as a horizontal unit. As on a surveyor's map or a scale drawing, line segments that are supposed to have the same length will look as if they do, and angles that are supposed to be equal will look equal. Some of the geometric results we obtain later, such as the relationship between the slopes of perpendicular lines, are valid only if equal scales are used on the two axes.

Computer and calculator displays are another matter. The vertical and horizontal scales on machine-generated graphs usually differ, with resulting distortions in distances, slopes, and angles. Circles may appear elliptical, and squares may appear rectangular or even as parallelograms. Right angles may appear as acute or obtuse. Circumstances like these require us to take extra care in interpreting what we see. High-quality computer software for drawing Cartesian graphs usually allows the user to compensate for such scale problems by adjusting the *aspect ratio* (the ratio of vertical to horizontal scale). Some computer screens also allow adjustment within a narrow range. When using graphing software, try to adjust your particular software/hardware configuration so that the horizontal and vertical diameters of a drawn circle appear to be equal.

## Increments and Distances

When a particle moves from one point to another, the net changes in its coordinates are called increments. They are calculated by subtracting the coordinates of the starting point from the coordinates of the ending point. An **increment** in a variable is the net change in the value of the variable. If $x$ changes from $x_1$ to $x_2$, then the increment in $x$ is $\Delta x = x_2 - x_1$.

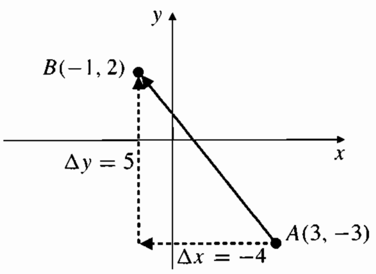



**Figure P.11** Increments in $x$ and $y$

**EXAMPLE 1** Find the increments in the coordinates of a particle that moves from $A(3, -3)$ to $B(-1, 2)$.

***Solution*** The increments (see Figure P.11) are:

$$\Delta x = -1 - 3 = -4 \qquad \text{and} \qquad \Delta y = 2 - (-3) = 5.$$

If $P(x_1, y_1)$ and $Q(x_2, y_2)$ are two points in the plane, the straight line segment $PQ$ is the hypotenuse of a right triangle $PCQ$, as shown in Figure P.12. The sides $PC$ and $CQ$ of the triangle have lengths

$$|\Delta x| = |x_2 - x_1| \qquad \text{and} \qquad |\Delta y| = |y_2 - y_1|.$$

These are the *horizontal distance* and *vertical distance* between $P$ and $Q$. By the Pythagorean Theorem, the length of $PQ$ is the square root of the sum of the squares of these lengths.

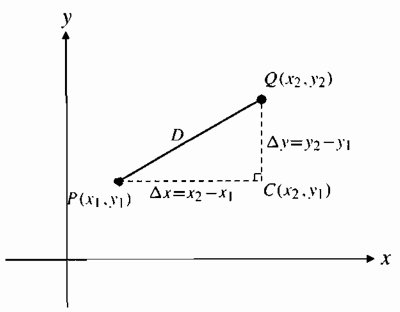



**Figure P.12** The distance from $P$ to $Q$ is $D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

**Distance formula for points in the plane**

The distance $D$ between $P(x_1, y_1)$ and $Q(x_2, y_2)$ is

$$D = \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

**EXAMPLE 2** The distance between $A(3, -3)$ and $B(-1, 2)$ in Figure P.11 is

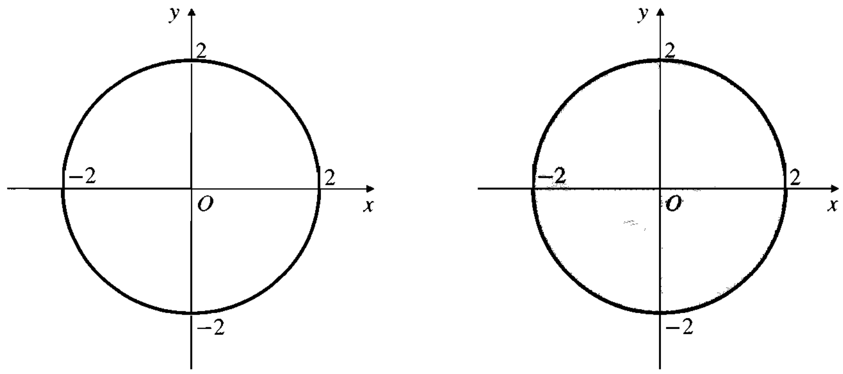
$$\sqrt{(-1 - 3)^2 + (2 - (-3))^2} = \sqrt{(-4)^2 + 5^2} = \sqrt{41} \text{ units.}$$

**EXAMPLE 3** The distance from the origin $O(0, 0)$ to a point $P(x, y)$ is

$$\sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2 + y^2}.$$

## Graphs

The **graph** of an equation (or inequality) involving the variables $x$ and $y$ is the set of all points $P(x, y)$ whose coordinates satisfy the equation (or inequality).




(a) (b)

**Figure P.13**
(a) The circle $x^2 + y^2 = 4$
(b) The disk $x^2 + y^2 \le 4$

**EXAMPLE 4** The equation $x^2 + y^2 = 4$ represents all points $P(x, y)$ whose distance from the origin is $\sqrt{x^2 + y^2} = \sqrt{4} = 2$. These points lie on the **circle** of radius 2 centred at the origin. This circle is the graph of the equation $x^2 + y^2 = 4$. (See Figure P.13(a).)

**EXAMPLE 5** Points $(x, y)$ whose coordinates satisfy the inequality $x^2 + y^2 \le 4$ all have distance $\le 2$ from the origin. The graph of the inequality is therefore the disk of radius 2 centred at the origin. (See Figure P.13(b).)

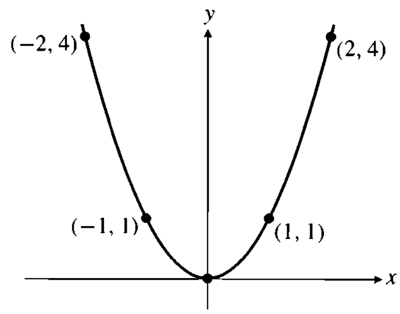



**Figure P.14** The parabola $y = x^2$

**EXAMPLE 6** Consider the equation $y = x^2$. Some points whose coordinates satisfy this equation are $(0, 0)$, $(1, 1)$, $(-1, 1)$, $(2, 4)$, and $(-2, 4)$. These points (and all others satisfying the equation) lie on a smooth curve called a **parabola**. (See Figure P.14.)

## Straight Lines

Given two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ in the plane, we call the increments $\Delta x = x_2 - x_1$ and $\Delta y = y_2 - y_1$, respectively, the **run** and the **rise** between $P_1$ and $P_2$. Two such points always determine a unique **straight line** (usually called simply a **line**) passing through them both. We call the line $P_1P_2$.

Any nonvertical line in the plane has the property that the ratio

$$m = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

has the *same value* for every choice of two distinct points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ on the line. (See Figure P.15.) The constant $m = \Delta y/\Delta x$ is called the **slope** of the nonvertical line.

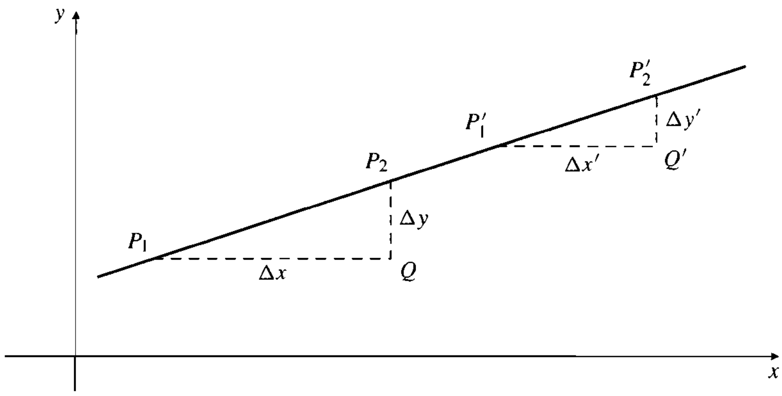



**Figure P.15** $\Delta y/\Delta x = \Delta y'/\Delta x'$ because triangles $P_1QP_2$ and $P_1'Q'P_2'$ are similar

**EXAMPLE 7** The slope of the line joining $A\,(3, -3)$ and $B\,(-1, 2)$ is

$$m = \frac{\Delta y}{\Delta x} = \frac{2-(-3)}{-1-3} = \frac{5}{-4} = -\frac{5}{4}.$$

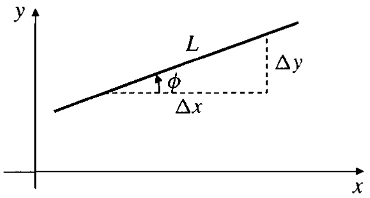



**Figure P.16** Line $L$ has inclination $\phi$

The slope tells us the direction and steepness of a line. A line with positive slope rises uphill to the right; one with negative slope falls downhill to the right. The greater the absolute value of the slope, the steeper the rise or fall. Since the run $\Delta x$ is zero for a vertical line, we cannot form the ratio $m$; the slope of a vertical line is *undefined.*

The direction of a line can also be measured by an angle. The **inclination** of a line is the smallest counterclockwise angle from the positive direction of the $x$-axis to the line. In Figure P.16 the angle $\phi$ (the Greek letter "phi") is the inclination of the line $L$. The inclination $\phi$ of any line satisfies $0° \le \phi < 180°$. The inclination of a horizontal line is $0°$ and that of a vertical line is $90°$.

Provided equal scales are used on the coordinate axes, the relationship between the slope $m$ of a nonvertical line and its inclination $\phi$ is shown in Figure P.16:

$$m = \frac{\Delta y}{\Delta x} = \tan\phi.$$

(The trigonometric function tan is defined in Section P.7.)

Parallel lines have the same inclination. If they are not vertical, they must therefore have the same slope. Conversely, lines with equal slopes have the same inclination and so are parallel.

If two nonvertical lines, $L_1$ and $L_2$, are perpendicular, their slopes $m_1$ and $m_2$ satisfy $m_1m_2 = -1$, so each slope is the *negative reciprocal* of the other:

$$m_1 = -\frac{1}{m_2} \quad \text{and} \quad m_2 = -\frac{1}{m_1}.$$

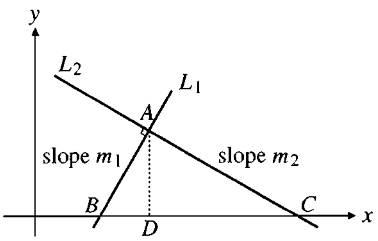



**Figure P.17** $\triangle ABD$ is similar to $\triangle CAD$

(This result also assumes equal scales on the two coordinate axes.) To see this, observe in Figure P.17 that

$$m_1 = \frac{AD}{BD} \quad \text{and} \quad m_2 = -\frac{AD}{DC}.$$

Since $\triangle ABD$ is similar to $\triangle CAD$, we have $\dfrac{AD}{BD} = \dfrac{DC}{AD}$, and so

$$m_1m_2 = \left(\frac{DC}{AD}\right)\left(-\frac{AD}{DC}\right) = -1.$$

## Equations of Lines

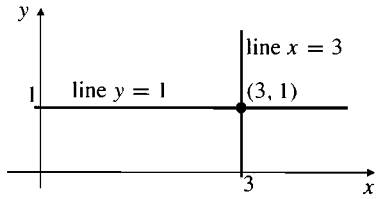


**Figure P.18** The lines $y = 1$ and $x = 3$

Straight lines are particularly simple graphs, and their corresponding equations are also simple. All points on the vertical line through the point $a$ on the $x$-axis have their $x$-coordinates equal to $a$. Thus $x = a$ is the equation of the line. Similarly, $y = b$ is the equation of the horizontal line meeting the $y$-axis at $b$.

**EXAMPLE 8** The horizontal and vertical lines passing through the point $(3, 1)$ (Figure P.18) have equations $y = 1$ and $x = 3$, respectively.

To write an equation for a nonvertical straight line $L$, it is enough to know its slope $m$ and the coordinates of one point $P_1(x_1, y_1)$ on it. If $P(x, y)$ is any other point on $L$, then

$$\frac{y - y_1}{x - x_1} = m,$$

so that

$$y - y_1 = m(x - x_1) \qquad \text{or} \qquad y = m(x - x_1) + y_1.$$

The equation

$$y = m(x - x_1) + y_1$$

is the **point-slope equation** of the line that passes through the point $(x_1, y_1)$ and has slope $m$.

**EXAMPLE 9** Find an equation of the line of slope $-2$ through the point $(1, 4)$.

***Solution*** We substitute $x_1 = 1$, $y_1 = 4$, and $m = -2$ into the point-slope form of the equation and obtain

$$y = -2(x - 1) + 4 \qquad \text{or} \qquad y = -2x + 6.$$

**EXAMPLE 10** Find an equation of the line through the points $(1, -1)$ and $(3, 5)$.

***Solution*** The slope of the line is $m = \dfrac{5 - (-1)}{3 - 1} = 3$. We can use this slope with either of the two points to write an equation of the line. If we use $(1, -1)$ we get

$$y = 3(x - 1) - 1, \qquad \text{which simplifies to} \qquad y = 3x - 4.$$

If we use $(3, 5)$ we get

$$y = 3(x - 3) + 5, \qquad \text{which also simplifies to} \qquad y = 3x - 4.$$

Either way, $y = 3x - 4$ is an equation of the line.

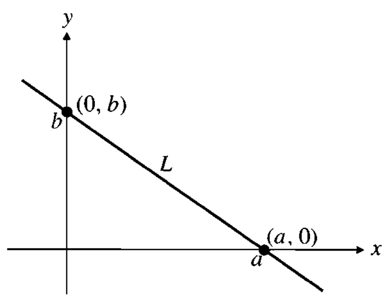


**Figure P.19** Line $L$ has $x$-intercept $a$ and $y$-intercept $b$

The $y$-coordinate of the point where a nonvertical line intersects the $y$-axis is called the ***y*-intercept** of the line. (See Figure P.19.) Similarly, the ***x*-intercept** of a nonhorizontal line is the $x$-coordinate of the point where it crosses the $x$-axis. A line with slope $m$ and $y$-intercept $b$ passes through the point $(0, b)$, so its equation is

$$y = m(x - 0) + b \qquad \text{or, more simply,} \qquad y = mx + b.$$

A line with slope $m$ and $x$-intercept $a$ passes through $(a, 0)$, and so its equation is

$$y = m(x - a).$$

The equation $y = mx + b$ is called the **slope–$y$-intercept equation** of the line with slope $m$ and $y$-intercept $b$.

The equation $y = m(x - a)$ is called the **slope–$x$-intercept equation** of the line with slope $m$ and $x$-intercept $a$.

**EXAMPLE 11** Find the slope and the two intercepts of the line with equation $8x + 5y = 20$.

***Solution*** Solving the equation for $y$ we get

$$y = \frac{20 - 8x}{5} = -\frac{8}{5}x + 4.$$

Comparing this with the general form $y = mx + b$ of the slope–$y$-intercept equation, we see that the slope of the line is $m = -8/5$, and the $y$-intercept is $b = 4$.

To find the $x$-intercept put $y = 0$ and solve for $x$, obtaining $8x = 20$, or $x = 5/2$. The $x$-intercept is $a = 5/2$.

The equation $Ax + By = C$ (where $A$ and $B$ are not both zero) is called the **general linear equation** in $x$ and $y$ because its graph always represents a straight line, and every line has an equation in this form.

Many important quantities are related by linear equations. Once we know that a relationship between two variables is linear, we can find it from any two pairs of corresponding values, just as we find the equation of a line from the coordinates of two points.

**EXAMPLE 12** The relationship between Fahrenheit temperature ($F$) and Celsius temperature ($C$) is given by a linear equation of the form $F = mC + b$. The freezing point of water is $F = 32°$ or $C = 0°$, while the boiling point is $F = 212°$ or $C = 100°$. Thus

$$32 = 0m + b \qquad \text{and} \qquad 212 = 100m + b,$$

so $b = 32$ and $m = (212 - 32)/100 = 9/5$. The relationship is given by the linear equation

$$F = \frac{9}{5}C + 32 \qquad \text{or} \qquad C = \frac{5}{9}(F - 32).$$

## EXERCISES P.2

In Exercises 1–4, a particle moves from $A$ to $B$. Find the net increments $\Delta x$ and $\Delta y$ in the particle's coordinates. Also find the distance from $A$ to $B$.

**1.** $A(0, 3)$, $B(4, 0)$

**2.** $A(-1, 2)$, $B(4, -10)$

**3.** $A(3, 2)$, $B(-1, -2)$

**4.** $A(0.5, 3)$, $B(2, 3)$

**5.** A particle starts at $A(-2, 3)$ and its coordinates change by $\Delta x = 4$ and $\Delta y = -7$. Find its new position.

**6.** A particle arrives at the point $(-2, -2)$ after its coordinates experience increments $\Delta x = -5$ and $\Delta y = 1$. From where did it start?

Describe the graphs of the equations and inequalities in Exercises 7–12.

**7.** $x^2 + y^2 = 1$

**8.** $x^2 + y^2 = 2$

**9.** $x^2 + y^2 \leq 1$

**10.** $x^2 + y^2 = 0$

**11.** $y \geq x^2$ **12.** $y < x^2$

In Exercises 13–14, find an equation for (a) the vertical line and (b) the horizontal line through the given point.

**13.** $(-2, 5/3)$ **14.** $(\sqrt{2}, -1.3)$

In Exercises 15–18, write an equation for the line through $P$ with slope $m$.

**15.** $P(-1, 1), \quad m = 1$ **16.** $P(-2, 2), \quad m = 1/2$

**17.** $P(0, b), \quad m = 2$ **18.** $P(a, 0), \quad m = -2$

In Exercises 19–20, does the given point $P$ lie on, above, or below the given line?

**19.** $P(2, 1), \quad 2x + 3y = 6$ **20.** $P(3, -1), \quad x - 4y = 7$

In Exercises 21–24, write an equation for the line through the two points.

**21.** $(0, 0), \quad (2, 3)$ **22.** $(-2, 1), \quad (2, -2)$

**23.** $(4, 1), \quad (-2, 3)$ **24.** $(-2, 0), \quad (0, 2)$

In Exercises 25–26, write an equation for the line with slope $m$ and $y$-intercept $b$.

**25.** $m = -2, \quad b = \sqrt{2}$ **26.** $m = -1/2, \quad b = -3$

In Exercises 27–30, determine the $x$- and $y$-intercepts and the slope of the given lines, and sketch their graphs.

**27.** $3x + 4y = 12$ **28.** $x + 2y = -4$

**29.** $\sqrt{2}x - \sqrt{3}y = 2$ **30.** $1.5x - 2y = -3$

In Exercises 31–32, find equations for the lines through $P$ that are (a) parallel to and (b) perpendicular to the given line.

**31.** $P(2, 1), \quad y = x + 2$ **32.** $P(-2, 2), \quad 2x + y = 4$

**33.** Find the point of intersection of the lines $3x + 4y = -6$ and $2x - 3y = 13$.

**34.** Find the point of intersection of the lines $2x + y = 8$ and $5x - 7y = 1$.

**35.** **(Two-intercept equations)** If a line is neither horizontal nor vertical and does not pass through the origin, show that its equation can be written in the form $\dfrac{x}{a} + \dfrac{y}{b} = 1$, where $a$ is its $x$-intercept and $b$ is its $y$-intercept.

**36.** Determine the intercepts and sketch the graph of the line $\dfrac{x}{2} - \dfrac{y}{3} = 1$.

**37.** Find the $y$-intercept of the line through the points $(2, 1)$ and $(3, -1)$.

**38.** A line passes through $(-2, 5)$ and $(k, 1)$ and has $x$-intercept 3. Find $k$.

**39.** The cost of printing $x$ copies of a pamphlet is $\$C$, where $C = Ax + B$ for certain constants $A$ and $B$. If it costs \$5,000 to print 10,000 copies and \$6,000 to print 15,000 copies, how much will it cost to print 100,000 copies?

**40.** **(Fahrenheit versus Celsius)** In the $FC$-plane, sketch the graph of the equation $C = \dfrac{5}{9}(F - 32)$ linking Fahrenheit and Celsius temperatures found in Example 12. On the same graph sketch the line with equation $C = F$. Is there a temperature at which a Celsius thermometer gives the same numerical reading as a Fahrenheit thermometer? If so, find that temperature.

### Geometry

**41.** By calculating the lengths of its three sides, show that the triangle with vertices at the points $A(2, 1)$, $B(6, 4)$, and $C(5, -3)$ is isosceles.

**42.** Show that the triangle with vertices $A(0, 0)$, $B(1, \sqrt{3})$, and $C(2, 0)$ is equilateral.

**43.** Show that the points $A(2, -1)$, $B(1, 3)$, and $C(-3, 2)$ are three vertices of a square and find the fourth vertex.

**44.** Find the coordinates of the midpoint on the line segment $P_1P_2$ joining the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$.

**45.** Find the coordinates of the point of the line segment joining the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ that is two-thirds of the way from $P_1$ to $P_2$.

**46.** The point $P$ lies on the $x$-axis and the point $Q$ lies on the line $y = -2x$. The point $(2, 1)$ is the midpoint of $PQ$. Find the coordinates of $P$.

In Exercises 47–48, interpret the equation as a statement about distances, and hence determine the graph of the equation.

**47.** $\sqrt{(x-2)^2 + y^2} = 4$

**48.** $\sqrt{(x-2)^2 + y^2} = \sqrt{x^2 + (y-2)^2}$

**49.** For what value of $k$ is the line $2x + ky = 3$ perpendicular to the line $4x + y = 1$? For what value of $k$ are the lines parallel?

**50.** Find the line that passes through the point $(1, 2)$ and through the point of intersection of the two lines $x + 2y = 3$ and $2x - 3y = -1$.

# P.3 Graphs of Quadratic Equations

This section reviews circles, parabolas, ellipses, and hyperbolas, the graphs that are represented by quadratic equations in two variables.

## Circles and Disks

The **circle** having **centre** $C$ and **radius** $a$ is the set of all points in the plane that are at distance $a$ from the point $C$.

The distance from $P(x, y)$ to the point $C(h, k)$ is $\sqrt{(x-h)^2 + (y-k)^2}$, so that the equation of the circle of radius $a > 0$ with centre at $C(h, k)$ is

$$\sqrt{(x-h)^2 + (y-k)^2} = a.$$

A simpler form of this equation is obtained by squaring both sides.

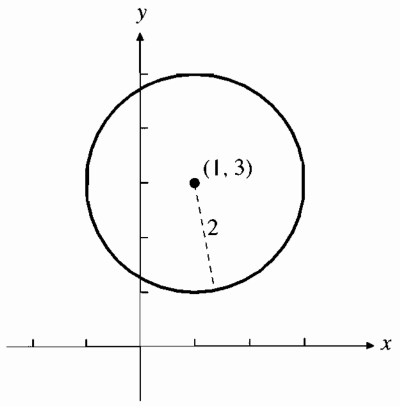



Figure P.20 Circle $(x-1)^2 + (y-3)^2 = 4$

**Standard equation of a circle**

The circle with centre $(h, k)$ and radius $a > 0$ has equation

$$(x-h)^2 + (y-k)^2 = a^2.$$

In particular, the circle with centre at the origin $(0, 0)$ and radius $a$ has equation

$$x^2 + y^2 = a^2.$$

**EXAMPLE 1** The circle with radius 2 and centre $(1, 3)$ (Figure P.20) has equation $(x-1)^2 + (y-3)^2 = 4$.

**EXAMPLE 2** The circle having equation $(x+2)^2 + (y-1)^2 = 7$ has centre at the point $(-2, 1)$ and radius $\sqrt{7}$. (See Figure P.21.)

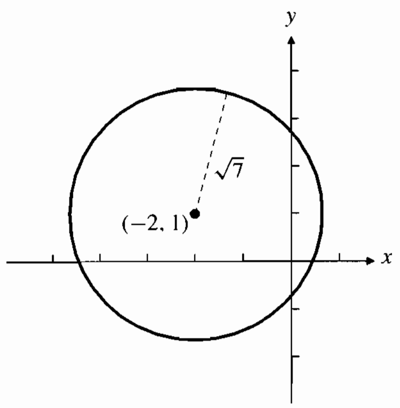



Figure P.21 Circle $(x+2)^2 + (y-1)^2 = 7$

If the squares in the standard equation $(x-h)^2 + (y-k)^2 = a^2$ are multiplied out, and all constant terms collected on the right-hand side, the equation becomes

$$x^2 - 2hx + y^2 - 2ky = a^2 - h^2 - k^2.$$

A quadratic equation of the form

$$x^2 + y^2 + 2ax + 2by = c$$

must represent a circle, a single point, or no points at all. To identify the graph, we complete the squares on the left side of the equation. Since $x^2 + 2ax$ are the first two terms of the square $(x+a)^2 = x^2 + 2ax + a^2$, we add $a^2$ to both sides to complete the square of the $x$ terms. (Note that $a^2$ is *the square of half the coefficient of* $x$.) Similarly, add $b^2$ to both sides to complete the square of the $y$ terms. The equation then becomes

$$(x+a)^2 + (y+b)^2 = c + a^2 + b^2.$$

If $c + a^2 + b^2 > 0$, the graph is a circle with centre $(-a, -b)$ and radius $\sqrt{c + a^2 + b^2}$. If $c + a^2 + b^2 = 0$, the graph consists of the single point $(-a, -b)$. If $c + a^2 + b^2 < 0$, no points lie on the graph.

**EXAMPLE 3** Find the centre and radius of the circle $x^2 + y^2 - 4x + 6y = 3$

***Solution*** Observe that $x^2 - 4x$ are the first two terms of the binomial square $(x-2)^2 = x^2 - 4x + 4$, and $y^2 + 6y$ are the first two terms of the square $(y+3)^2 = y^2 + 6y + 9$. Hence we add $4 + 9$ to both sides of the given equation and obtain

$$x^2 - 4x + 4 + y^2 + 6y + 9 = 3 + 4 + 9 \quad \text{or} \quad (x-2)^2 + (y+3)^2 = 16.$$

This is the equation of a circle with centre $(2, -3)$ and radius 4.

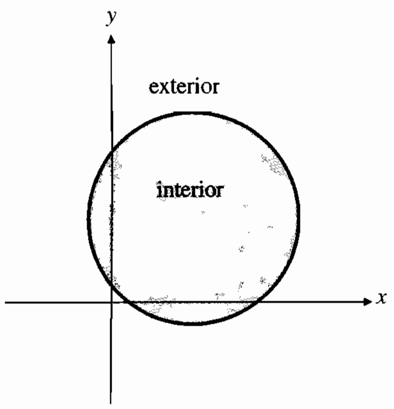



Figure P.22 The interior of a circle (darkly shaded) and the exterior (lightly shaded)

The set of all points *inside* a circle is called the **interior** of the circle; it is also called an **open disk**. The set of all points *outside* the circle is called the **exterior** of the circle. (See Figure P.22.) The interior of a circle together with the circle itself is called a **closed disk**, or simply a **disk**. The inequality

$$(x-h)^2 + (y-k)^2 \le a^2$$

represents the disk of radius $|a|$ centred at $(h, k)$.

**EXAMPLE 4** Identify the graphs of:

(a) $x^2 + 2x + y^2 \le 8$ (b) $x^2 + 2x + y^2 < 8$ (c) $x^2 + 2x + y^2 > 8$.

***Solution*** We can complete the square in the equation $x^2 + y^2 + 2x = 8$ as follows:

$$x^2 + 2x + 1 + y^2 = 8 + 1$$
$$(x + 1)^2 + y^2 = 9.$$

Thus the equation represents the circle of radius 3 with centre at $(-1, 0)$. Inequality (a) represents the (closed) disk with the same radius and centre. (See Figure P.23.) Inequality (b) represents the interior of the circle (or the open disk). Inequality (c) represents the exterior of the circle.

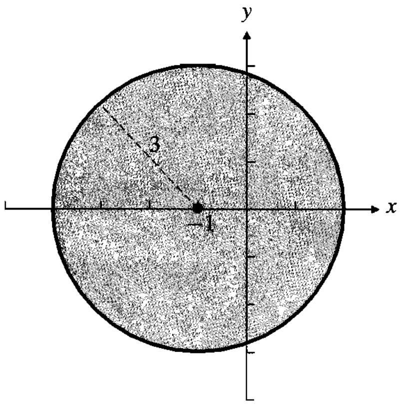



**Figure P.23** The disk $x^2 + y^2 + 2x \le 8$

## Equations of Parabolas

A **parabola** is a plane curve whose points are equidistant from a fixed point $F$ and a fixed straight line $L$ that does not pass through $F$. The point $F$ is the **focus** of the parabola; the line $L$ is the parabola's **directrix**. The line through $F$ perpendicular to $L$ is the parabola's **axis**. The point $V$ where the axis meets the parabola is the parabola's **vertex**.

Observe that the vertex $V$ of a parabola is halfway between the focus $F$ and the point on the directrix $L$ that is closest to $F$. If the directrix is either horizontal or vertical, and the vertex is at the origin, then the parabola will have a particularly simple equation.

**EXAMPLE 5** Find an equation of the parabola having the point $F(0, p)$ as focus and the line $L$ with equation $y = -p$ as directrix.

***Solution*** If $P(x, y)$ is any point on the parabola, then (see Figure P.24) the distances from $P$ to $F$ and to (the closest point $Q$ on) the line $L$ are given by

$$PF = \sqrt{(x-0)^2 + (y-p)^2} = \sqrt{x^2 + y^2 - 2py + p^2}$$
$$PQ = \sqrt{(x-x)^2 + (y-(-p))^2} = \sqrt{y^2 + 2py + p^2}.$$

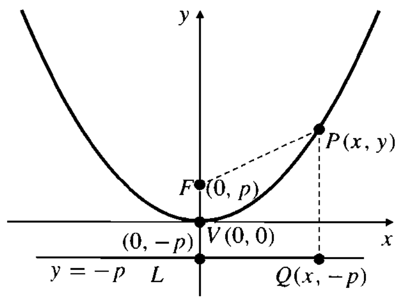



**Figure P.24** The parabola $4py = x^2$ with focus $F(0, p)$ and directrix $y = -p$

Since $P$ is on the parabola, $PF = PQ$ and so the squares of these distances are also equal:

$$x^2 + y^2 - 2py + p^2 = y^2 + 2py + p^2,$$

or, after simplifying,

$$x^2 = 4py \qquad \text{or} \qquad y = \frac{x^2}{4p} \qquad \text{(called \textbf{standard forms})}.$$

Figure P.24 shows the situation for $p > 0$; the parabola opens upward and is symmetric about its axis, the $y$-axis. If $p < 0$, the focus $(0, p)$ will lie below the origin and the directrix $y = -p$ will lie above the origin. In this case the parabola will open downward instead of upward.

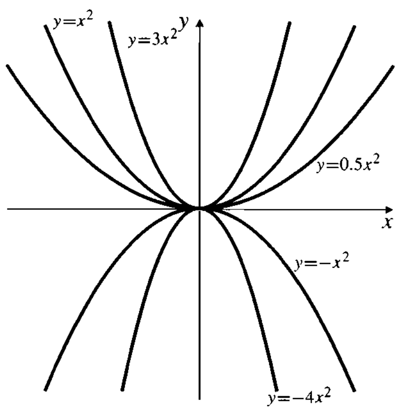



**Figure P.25** Some parabolas $y = ax^2$

Figure P.25 shows several parabolas with equations of the form $y = ax^2$ for positive and negative values of $a$.

**EXAMPLE 6** An equation for the parabola with focus $(0, 1)$ and directrix $y = -1$ is $y = x^2/4$, or $x^2 = 4y$. (We took $p = 1$ in the standard equation.)

**EXAMPLE 7** Find the focus and directrix of the parabola $y = -x^2$.

***Solution*** The given equation matches the standard form $y = x^2/(4p)$ provided $4p = -1$. Thus $p = -1/4$. The focus is $(0, -1/4)$, and the directrix is the line $y = 1/4$.

Interchanging the roles of $x$ and $y$ in the derivation of the standard equation above shows that the equation

$$y^2 = 4px \qquad \text{or} \qquad x = \frac{y^2}{4p} \qquad \text{(standard equation)}$$

represents a parabola with focus at $(p, 0)$ and vertical directrix $x = -p$. The axis is the $x$-axis.

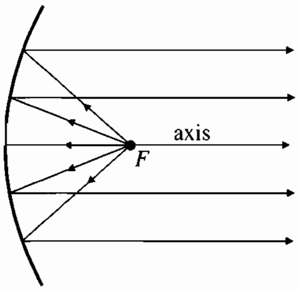



**Figure P.26** Reflection by a parabola

## Reflective Properties of Parabolas

One of the chief applications of parabolas is their use as reflectors of light and radio waves. Rays originating from the focus of a parabola will be reflected in a beam parallel to the axis, as shown in Figure P.26. Similarly, all the rays in a beam striking a parabola parallel to its axis will reflect through the focus. This property is the reason why telescopes and spotlights use parabolic mirrors and radio telescopes and microwave antennas are parabolic in shape. We will examine this property of parabolas more carefully in Section 8.1.

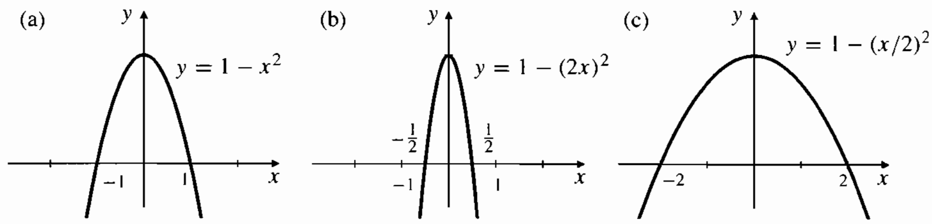



**Figure P.27** Horizontal scaling:
(a) the graph $y = 1 - x^2$
(b) graph of (a) compressed horizontally
(c) graph of (a) expanded horizontally

## Scaling a Graph

The graph of an equation can be compressed or expanded horizontally by replacing $x$ with a multiple of $x$. If $a$ is a positive number, replacing $x$ with $ax$ in an equation multiplies horizontal distances in the graph of the equation by a factor $1/a$. (See Figure P.27.) Replacing $y$ with $ay$ will multiply vertical distances in a similar way.

You may find it surprising that, like circles, all parabolas are *similar* geometric figures; they may have different sizes, but they all have the same shape. We can change the *size* while preserving the shape of a curve represented by an equation in $x$ and $y$ by scaling both the coordinates by the same amount. If we scale the equation $4py = x^2$ by replacing $x$ and $y$ with $4px$ and $4py$, respectively, we get $4p(4py) = (4px)^2$, or $y = x^2$. Thus the general parabola $4py = x^2$ has the same shape as the specific parabola $y = x^2$, as shown in Figure P.28.

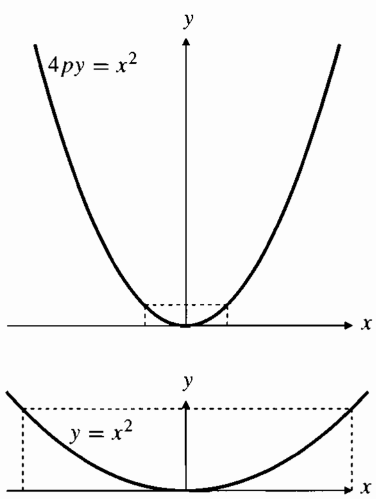



**Figure P.28** The two parabolas are similar. Compare the parts inside the rectangles

## Shifting a Graph

The graph of an equation (or inequality) can be shifted $c$ units horizontally by replacing $x$ with $x - c$ or vertically by replacing $y$ with $y - c$.

**Shifts**

To shift a graph $c$ units to the right, replace $x$ in its equation or inequality with $x - c$. (If $c < 0$, the shift will be to the left.)

To shift a graph $c$ units upward, replace $y$ in its equation or inequality with $y - c$. (If $c < 0$, the shift will be downward.)

**EXAMPLE 8** The graph of $y = (x-3)^2$ is the parabola $y = x^2$ shifted 3 units to the right. The graph of $y = (x+1)^2$ is the parabola $y = x^2$ shifted 1 unit to the left. (See Figure P.29(a).)

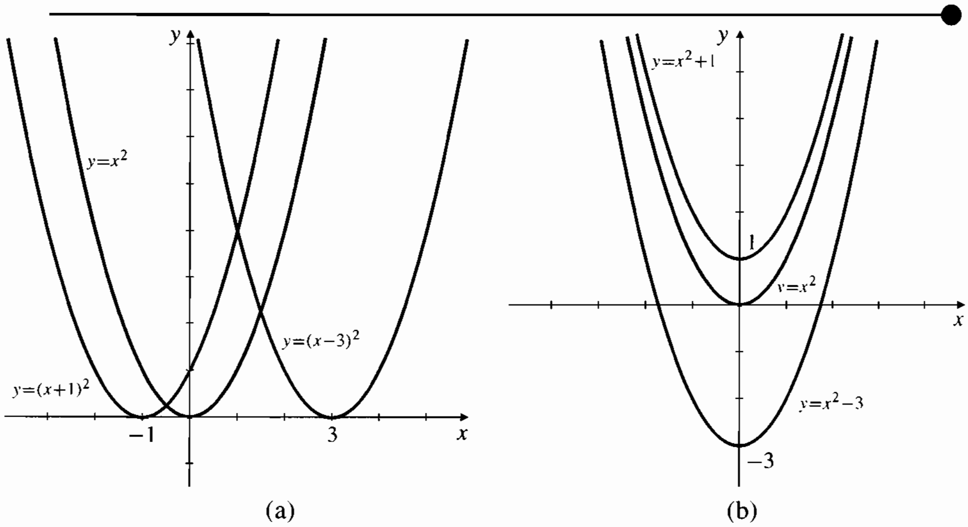



**Figure P.29**
(a) Horizontal shifts of $y = x^2$
(b) Vertical shifts of $y = x^2$

**EXAMPLE 9** The graph of $y = x^2 + 1$ (or $y - 1 = x^2$) is the parabola $y = x^2$ shifted upward 1 unit. The graph of $y = x^2 - 3$ (or $y - (-3) = x^2$), is the parabola $y = x^2$ shifted downward 3 units. (See Figure P.29(b).)

**EXAMPLE 10** The circle with equation $(x-h)^2 + (y-k)^2 = a^2$ having centre $(h, k)$ and radius $a$ can be obtained by shifting the circle $x^2 + y^2 = a^2$ of radius $a$ centred at the origin $h$ units to the right and $k$ units upward. These shifts correspond to replacing $x$ with $x - h$ and $y$ with $y - k$.

The graph of $y = ax^2 + bx + c$ is a parabola whose axis is parallel to the $y$-axis. The parabola opens upward if $a > 0$ and downward if $a < 0$. We can complete the square and write the equation in the form $y = a(x-h)^2 + k$ to find the vertex $(h, k)$.

**EXAMPLE 11** Describe the graph of $y = x^2 - 4x + 3$.

***Solution*** The equation $y = x^2 - 4x + 3$ represents a parabola, opening upward. To find its vertex and axis we can complete the square:

$$y = x^2 - 4x + 4 - 1 = (x-2)^2 - 1, \qquad \text{so} \quad y - (-1) = (x-2)^2.$$

This curve is the parabola $y = x^2$ shifted to the right 2 units and down 1 unit. Therefore, its vertex is $(2, -1)$, and its axis is the line $x = 2$. Since $y = x^2$ has focus $(0, 1/4)$, the focus of this parabola is $(0 + 2, (1/4) - 1)$, or $(2, -3/4)$. (See Figure P.30.)

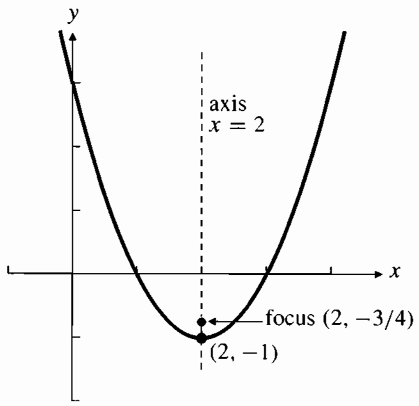



**Figure P.30** The parabola $y = x^2 - 4x + 3$

## Ellipses and Hyperbolas

If $a$ and $b$ are positive numbers, the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

represents a curve called an **ellipse** that lies wholly within the rectangle $-a \le x \le a$, $-b \le y \le b$. (Why?) If $a = b$, the ellipse is just the circle of radius $a$ centred at the origin. If $a \ne b$, the ellipse is a circle that has been squashed by scaling it by different amounts in the two coordinate directions.

The ellipse has centre at the origin, and it passes through the four points $(a, 0)$, $(0, b)$, $(-a, 0)$, and $(0, -b)$. (See Figure P.31.) The line segments from $(-a, 0)$ to $(a, 0)$ and from $(0, -b)$ to $(0, b)$ are called the **principal axes** of the ellipse; the longer of the two is the **major axis**, and the shorter is the **minor axis**.

**EXAMPLE 12** The equation $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$ represents an ellipse with major axis from $(-3, 0)$ to $(3, 0)$ and minor axis from $(0, -2)$ to $(0, 2)$.

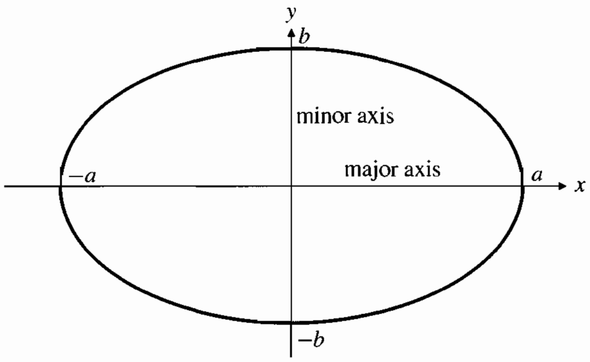



**Figure P.31** The ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

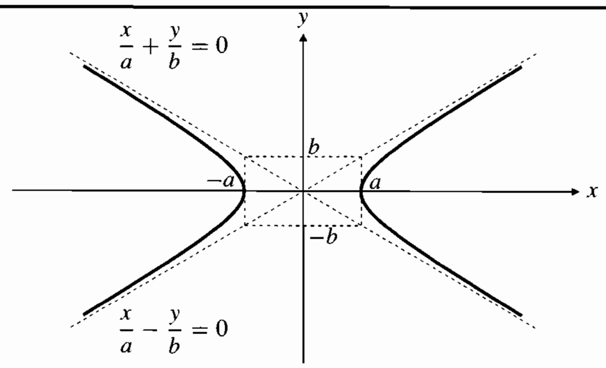



**Figure P.32** The hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ and its asymptotes

The equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

represents a curve called a **hyperbola** that has centre at the origin and passes through the points $(-a, 0)$ and $(a, 0)$. (See Figure P.32.) The curve is in two parts (called **branches**). Each branch approaches two straight lines (called **asymptotes**) as it recedes far away from the origin. The asymptotes have equations

$$\frac{x}{a} - \frac{y}{b} = 0 \qquad \text{and} \qquad \frac{x}{a} + \frac{y}{b} = 0.$$

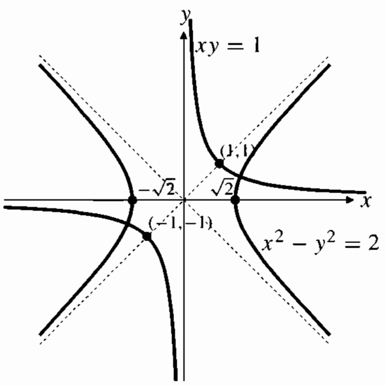



**Figure P.33** Two rectangular hyperbolas

The equation $xy = 1$ also represents a hyperbola. This one passes through the points $(-1, -1)$ and $(1, 1)$ and has the coordinate axes as its asymptotes. It is, in fact, the hyperbola $x^2 - y^2 = 2$ rotated 45° counterclockwise about the origin. (See Figure P.33.) These hyperbolas are called **rectangular hyperbolas**, since their asymptotes intersect at right angles.

We will study ellipses and hyperbolas in more detail in Chapter 8.

## EXERCISES P.3

In Exercises 1–4, write an equation for the circle with centre $C$ and radius $r$.

**1.** $C(0, 0)$, $r = 4$ **2.** $C(0, 2)$, $r = 2$

**3.** $C(-2, 0)$, $r = 3$ **4.** $C(3, -4)$, $r = 5$

In Exercises 5–8, find the centre and radius of the circle having the given equation.

**5.** $x^2 + y^2 - 2x = 3$ **6.** $x^2 + y^2 + 4y = 0$

**7.** $x^2 + y^2 - 2x + 4y = 4$ **8.** $x^2 + y^2 - 2x - y + 1 = 0$

Describe the regions defined by the inequalities and pairs of inequalities in Exercises 9–16.

**9.** $x^2 + y^2 > 1$ **10.** $x^2 + y^2 < 4$

**11.** $(x + 1)^2 + y^2 \le 4$ **12.** $x^2 + (y - 2)^2 \le 4$

**13.** $x^2 + y^2 > 1, \quad x^2 + y^2 < 4$

**14.** $x^2 + y^2 \le 4, \quad (x + 2)^2 + y^2 \le 4$

**15.** $x^2 + y^2 < 2x, \quad x^2 + y^2 < 2y$

**16.** $x^2 + y^2 - 4x + 2y > 4, \quad x + y > 1$

**17.** Write an inequality that describes the interior of the circle with centre $(-1, 2)$ and radius $\sqrt{6}$.

**18.** Write an inequality that describes the exterior of the circle with centre $(2, -3)$ and radius 4.

**19.** Write a pair of inequalities that describe that part of the interior of the circle with centre $(0, 0)$ and radius $\sqrt{2}$ lying on or to the right of the vertical line through $(1, 0)$.

**20.** Write a pair of inequalities that describe the points that lie outside the circle with centre $(0, 0)$ and radius 2, and inside the circle with centre $(1, 3)$ that passes through the origin.

In Exercises 21–24, write an equation of the parabola having the given focus and directrix.

**21.** Focus: $(0, 4)$ Directrix: $y = -4$

**22.** Focus: $(0, -1/2)$ Directrix: $y = 1/2$

**23.** Focus: $(2, 0)$ Directrix: $x = -2$

**24.** Focus: $(-1, 0)$ Directrix: $x = 1$

In Exercises 25–28, find the parabola's focus and directrix, and make a sketch showing the parabola, focus, and directrix.

**25.** $y = x^2/2$ **26.** $y = -x^2$

**27.** $x = -y^2/4$ **28.** $x = y^2/16$

**29.** Figure P.34 shows the graph $y = x^2$ and four shifted versions of it. Write equations for the shifted versions.

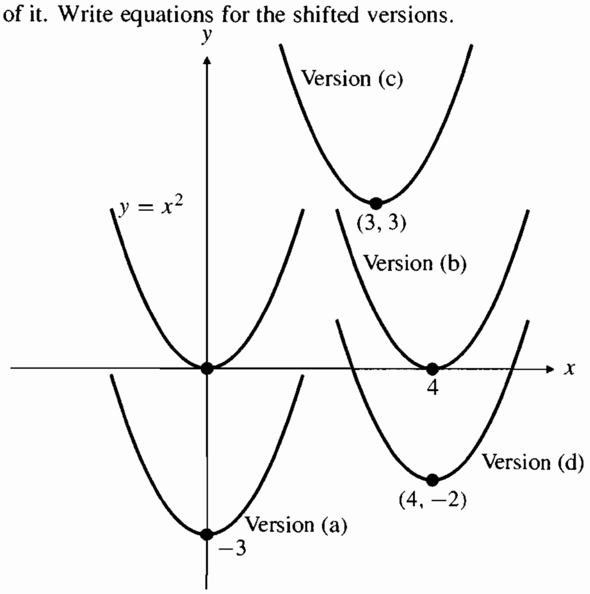



**Figure P.34**

**30.** What equations result from shifting the line $y = mx$
(a) horizontally to make it pass through the point $(a, b)$
(b) vertically to make it pass through $(a, b)$?

In Exercises 31–34, the graph of $y = \sqrt{x+1}$ is to be scaled in the indicated way. Give the equation of the graph that results from the scaling.

**31.** horizontal distances multiplied by 3

**32.** vertical distances divided by 4

**33.** horizontal distances multiplied by 2/3

**34.** horizontal distances divided by 4 and vertical distances multiplied by 2

In Exercises 35–38, write an equation for the graph obtained by shifting the graph of the given equation as indicated.

**35.** $y = 1 - x^2$ down 1, left 1

**36.** $x^2 + y^2 = 5$ up 2, left 4

**37.** $y = (x-1)^2 - 1$ down 1, right 1

**38.** $y = \sqrt{x}$ down 2, left 4

Find the points of intersection of the pairs of curves in Exercises 39–42.

**39.** $y = x^2 + 3, \quad y = 3x + 1$

**40.** $y = x^2 - 6, \quad y = 4x - x^2$

**41.** $x^2 + y^2 = 25, \quad 3x + 4y = 0$

**42.** $2x^2 + 2y^2 = 5, \quad xy = 1$

In Exercises 43–50, identify and sketch the curve represented by the given equation.

**43.** $\dfrac{x^2}{4} + y^2 = 1$ **44.** $9x^2 + 16y^2 = 144$

**45.** $\dfrac{(x-3)^2}{9} + \dfrac{(y+2)^2}{4} = 1$ **46.** $(x-1)^2 + \dfrac{(y+1)^2}{4} = 4$

**47.** $\dfrac{x^2}{4} - y^2 = 1$ **48.** $x^2 - y^2 = -1$

**49.** $xy = -4$ **50.** $(x-1)(y+2) = 1$

**51.** What is the effect on the graph of an equation in $x$ and $y$ of
(a) replacing $x$ with $-x$?
(b) replacing $y$ with $-y$?

**52.** What is the effect on the graph of an equation in $x$ and $y$ of replacing $x$ with $-x$ and $y$ with $-y$ simultaneously?

**53.** Sketch the graph of $|x| + |y| = 1$.

## P.4 Functions and Their Graphs

The area of a circle depends on its radius. The temperature at which water boils depends on the altitude above sea level. The interest paid on a cash investment depends on the length of time for which the investment is made.

Whenever one quantity depends on another quantity, we say that the former quantity is a function of the latter. For instance, the area $A$ of a circle depends on the radius $r$ according to the formula

$$A = \pi r^2,$$

so we say that the area is a function of the radius. The formula is a *rule* that tells us how to calculate a *unique* (single) output value of the area $A$ for each possible input value of the radius $r$.

The set of all possible input values for the radius is called the **domain** of the function. The set of all output values of the area is the **range** of the function. Since circles cannot have negative radii or areas, the domain and range of the circular area function are both the interval $[0, \infty)$ consisting of all nonnegative real numbers.

The domain and range of a mathematical function can be any sets of objects; they do not have to consist of numbers. Throughout much of this book, however, the domains and ranges of functions we consider will be sets of real numbers.

In calculus we often want to refer to a generic function without having any particular formula in mind. To denote that $y$ is a function of $x$ we write

$$y = f(x),$$

which we read as "$y$ equals $f$ of $x$." In this notation, due to eighteenth-century mathematician Leonhard Euler, the function is represented by the symbol $f$. Also, $x$, called the **independent variable**, represents an input value from the domain of $f$, and $y$, the **dependent variable**, represents the corresponding output value $f(x)$ in the range of $f$.

**DEFINITION 1**

> A **function** $f$ on a set $D$ into a set $S$ is a rule that assigns a *unique* element $f(x)$ in $S$ to each element $x$ in $D$.

In this definition $D = \mathcal{D}(f)$ (read "$D$ of $f$") is the domain of the function $f$. The range $\mathcal{R}(f)$ of $f$ is the subset of $S$ consisting of all *values* $f(x)$ of the function. Think of a function $f$ as a kind of machine (Figure P.35) that produces an output value $f(x)$ in its range whenever we feed it an input value $x$ from its domain.

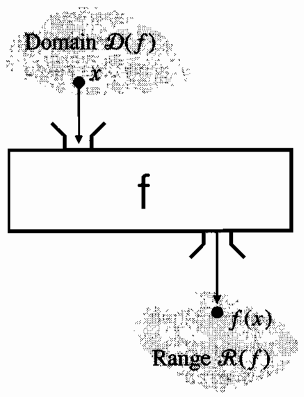


**Figure P.35** A function machine

There are several ways to represent a function symbolically. The squaring function that converts any input real number $x$ into its square $x^2$ can be denoted:

(a) by a formula such as $y = x^2$, which uses a dependent variable $y$ to denote the value of the function;

(b) by a formula such as $f(x) = x^2$, which defines a function symbol $f$ to name the function; or

(c) by a mapping rule such as $x \longrightarrow x^2$. (Read this as "$x$ goes to $x^2$.")

In this book we will usually use either (a) or (b) to define functions. Strictly speaking, we should call a function $f$ and not $f(x)$, since the latter denotes the value of the function at the point $x$. However, as is common usage, we will often refer to the function as $f(x)$ in order to name the variable on which $f$ depends. Sometimes it is convenient to use the same letter to denote both a dependent variable and a function symbol; the circular area function can be written $A = f(r) = \pi r^2$ or as $A = A(r) = \pi r^2$. In the latter case we are using $A$ to denote both the dependent variable and the name of the function.

**EXAMPLE 1** The volume of a ball of radius $r$ is given by the function

$$V(r) = \frac{4}{3}\pi r^3$$

for $r \geq 0$. Thus the volume of a ball of radius 3 ft is

$$V(3) = \frac{4}{3}\pi(3)^3 = 36\pi \text{ ft}^3.$$

Note how the variable $r$ is replaced by the special value 3 in the formula defining the function to obtain the value of the function at $r = 3$.

**EXAMPLE 2** A function $F$ is defined for all real numbers $t$ by

$$F(t) = 2t + 3.$$

Find the output values of $F$ that correspond to the input values 0, 2, $x + 2$, and $F(2)$.

***Solution*** In each case we substitute the given input for $t$ in the definition of $F$:

$$\begin{aligned} F(0) &= 2(0) + 3 = 0 + 3 = 3 \\ F(2) &= 2(2) + 3 = 4 + 3 = 7 \\ F(x+2) &= 2(x+2) + 3 = 2x + 7 \\ F(F(2)) &= F(7) = 2(7) + 3 = 17. \end{aligned}$$

## The Domain Convention

A function is not properly defined until its domain is specified. For instance, the function $f(x) = x^2$ defined for all real numbers $x \geq 0$ is different from the function $g(x) = x^2$ defined for all real $x$ because they have different domains, even though they have the same values at every point where both are defined. In Chapters 1–9 we will be dealing with real functions (functions whose input and output values are real numbers). When the domain of such a function is not specified explicitly, we will assume that the domain is the largest set of real numbers to which the function assigns real values. Thus, if we talk about the function $x^2$ without specifying a domain, we mean the function $g(x)$ above.

**The domain convention**

When a function $f$ is defined without specifying its domain, we assume that the domain consists of all real numbers $x$ for which the value $f(x)$ of the function is a real number.

In practice, it is often easy to determine the domain of a function $f(x)$ given by an explicit formula. We just have to exclude those values of $x$ that would result in dividing by 0 or taking even roots of negative numbers.

**EXAMPLE 3** **The square root function.** The domain of $f(x) = \sqrt{x}$ is the interval $[0, \infty)$, since negative numbers do not have real square roots. We have $f(0) = 0$, $f(4) = 2$, $f(10) \approx 3.16228$. Note that, although there are *two* numbers whose square is 4, namely, $-2$ and 2, only *one* of these numbers, 2, is the square root of 4. (Remember that a function assigns a *unique* value to each element in its domain; it cannot assign two different values to the same input.) The **square root function** $\sqrt{x}$ always denotes the *nonnegative* square root of $x$. The two solutions of the equation $x^2 = 4$ are $x = \sqrt{4} = 2$ and $x = -\sqrt{4} = -2$.

**EXAMPLE 4** The domain of the function $h(x) = \dfrac{x}{x^2 - 4}$ consists of all real numbers except $x = -2$ and $x = 2$. Expressed in terms of intervals,

$$\mathcal{D}(f) = (-\infty, -2) \cup (-2, 2) \cup (2, \infty).$$

Most of the functions we encounter will have domains that are either intervals or unions of intervals.

**EXAMPLE 5** The domain of $S(t) = \sqrt{1 - t^2}$ consists of all real numbers $t$ for which $1 - t^2 \geq 0$. Thus we require that $t^2 \leq 1$, or $-1 \leq t \leq 1$. The domain is the closed interval $[-1, 1]$.

## Graphs of Functions

An old maxim states that "a picture is worth a thousand words." This is certainly true in mathematics; the behaviour of a function is best described by drawing its graph.

The **graph of a function** $f$ is just the graph of the *equation* $y = f(x)$. It consists of those points in the Cartesian plane whose coordinates $(x, y)$ are pairs of input–output values for $f$. Thus $(x, y)$ lies on the graph of $f$ provided $x$ is in the domain of $f$ and $y = f(x)$.

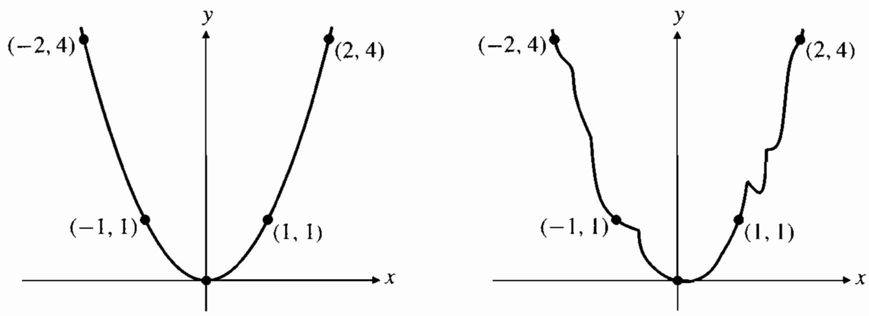
Drawing the graph of a function $f$ sometimes involves making a table of coordinate pairs $(x, f(x))$ for various values of $x$ in the domain of $f$, then plotting these points and connecting them with a "smooth curve."

**EXAMPLE 6** Graph the function $f(x) = x^2$.

Table 1.

| $x$ | $y = f(x)$ |
|---|---|
| $-2$ | 4 |
| $-1$ | 1 |
| 0 | 0 |
| 1 | 1 |
| 2 | 4 |

***Solution*** Make a table of $(x, y)$ pairs that satisfy $y = x^2$. (See Table 1.) Now plot the points and join them with a smooth curve. (See Figure P.36(a).)




(a) (b)

**Figure P.36**
(a) Correct graph of $f(x) = x^2$
(b) Incorrect graph of $f(x) = x^2$

How do we know the graph is smooth and doesn't do weird things between the points we have calculated, for example, as shown in Figure P.36(b)? We could, of course, plot more points, spaced more closely together, but how do we know how the graph behaves between the points we have plotted? In Chapter 4, calculus will provide useful tools for answering these questions.

Some functions occur often enough in applications that you should be familiar with their graphs. Some of these are shown in Figures P.37–P.46. Study them for a while; they are worth remembering. Note, in particular, the graph of the **absolute value function**, $f(x) = |x|$, shown in Figure P.46. It is made up of the two half-lines $y = -x$ for $x < 0$ and $y = x$ for $x \geq 0$.

If you know the effects of vertical and horizontal shifts on the equations representing graphs (see Section P.3), you can easily sketch some graphs that are shifted versions of the ones in Figures P.37–P.46.

**EXAMPLE 7** Sketch the graph of $y = 1 + \sqrt{x - 4}$.

***Solution*** This is just the graph of $y = \sqrt{x}$ in Figure P.40 shifted to the right 4 units (because $x$ is replaced by $x - 4$) and up 1 unit. See Figure P.47.

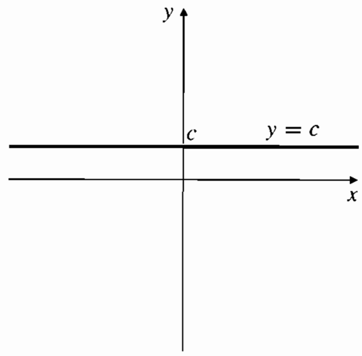



**Figure P.37** The graph of a constant function $f(x) = c$

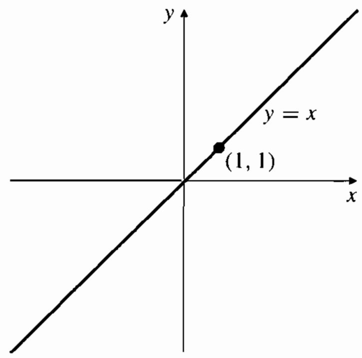



**Figure P.38** The graph of $f(x) = x$

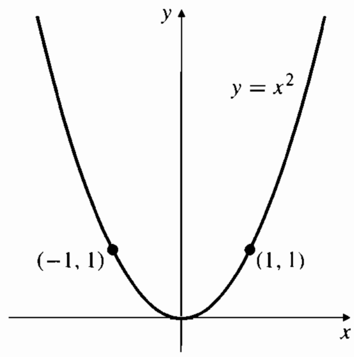



**Figure P.39** The graph of $f(x) = x^2$

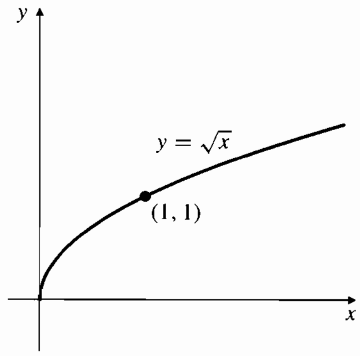



**Figure P.40** The graph of $f(x) = \sqrt{x}$

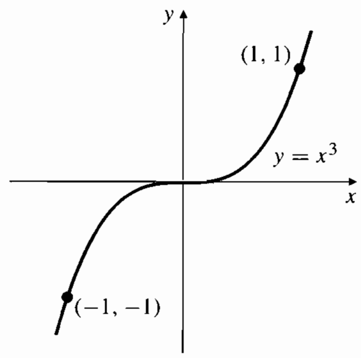



**Figure P.41** The graph of $f(x) = x^3$

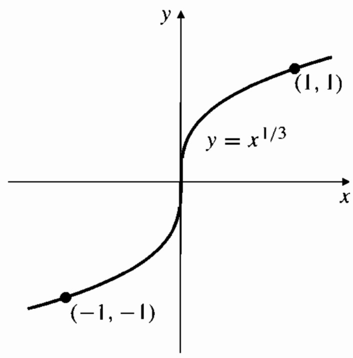



**Figure P.42** The graph of $f(x) = x^{1/3}$

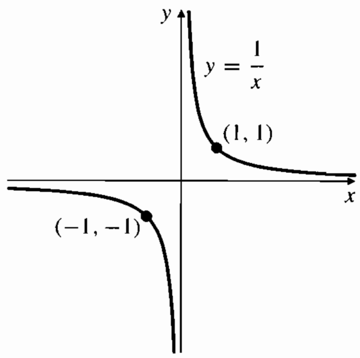



**Figure P.43** The graph of $f(x) = 1/x$

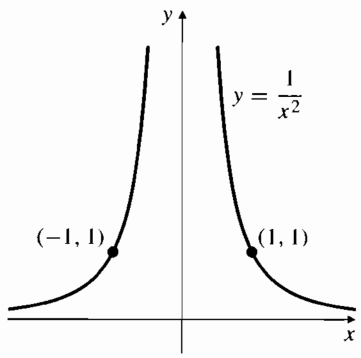



**Figure P.44** The graph of $f(x) = 1/x^2$

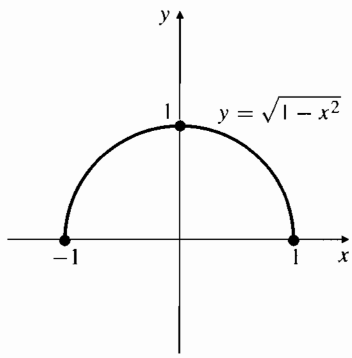



**Figure P.45** The graph of $f(x) = \sqrt{1 - x^2}$

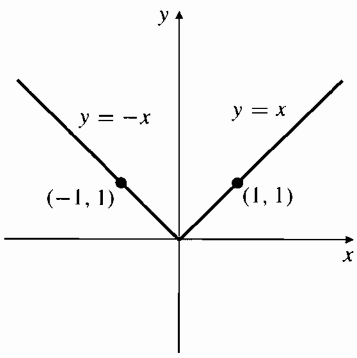



**Figure P.46** The graph of $f(x) = |x|$

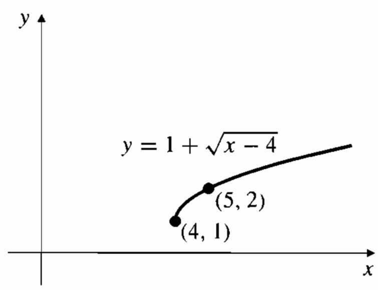



**Figure P.47** The graph of $y = \sqrt{x}$ shifted right 4 units and up 1 unit

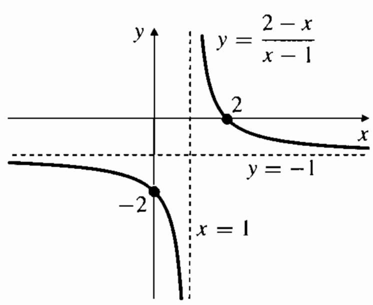



**Figure P.48** The graph of $\dfrac{2 - x}{x - 1}$

**EXAMPLE 8** Sketch the graph of the function $f(x) = \dfrac{2-x}{x-1}$.

***Solution*** It is not immediately obvious that this graph is a shifted version of a known graph. To see that it is, we can divide $x - 1$ into $2 - x$ to get a quotient of $-1$ and a remainder of 1:

$$\frac{2-x}{x-1} = \frac{-x+1+1}{x-1} = \frac{-(x-1)+1}{x-1} = -1 + \frac{1}{x-1}.$$

Thus, the graph is that of $1/x$ from Figure P.43 shifted to the right 1 unit and down 1 unit. See Figure P.48.

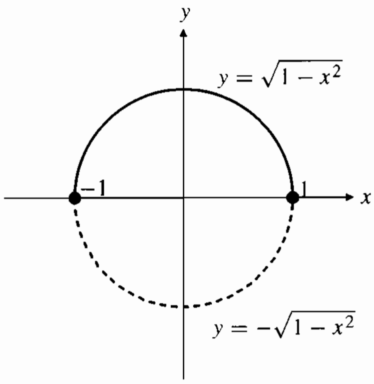



**Figure P.49** The circle $x^2 + y^2 = 1$ is not the graph of a function

Not every curve you can draw is the graph of a function. A function $f$ can have only one value $f(x)$ for each $x$ in its domain, so no *vertical line* can intersect the graph of a function at more than one point. If $a$ is in the domain of function $f$, then the vertical line $x = a$ will intersect the graph of $f$ at the single point $(a, f(a))$. The circle $x^2 + y^2 = 1$ in Figure P.49 cannot be the graph of a function since some vertical lines intersect it twice. It is, however, the union of the graphs of two functions, namely,

$$y = \sqrt{1-x^2} \qquad \text{and} \qquad y = -\sqrt{1-x^2},$$

which are, respectively, the upper and lower halves (semicircles) of the given circle.

## Even and Odd Functions; Symmetry and Reflections

It often happens that the graph of a function will have certain kinds of symmetry. The simplest kinds of symmetry relate the values of a function at $x$ and $-x$.

**DEFINITION 2**

**Even and odd functions**

Suppose that $-x$ belongs to the domain of $f$ whenever $x$ does. We say that $f$ is an **even function** if

$$f(-x) = f(x) \qquad \text{for every } x \text{ in the domain of } f.$$

We say that $f$ is an **odd function** if

$$f(-x) = -f(x) \qquad \text{for every } x \text{ in the domain of } f.$$

The names *even* and *odd* come from the fact that even powers such as $x^0 = 1, x^2, x^4, \ldots, x^{-2}, x^{-4}, \ldots$ are even functions, and odd powers such as $x^1 = x, x^3, \ldots, x^{-1}, x^{-3}, \ldots$ are odd functions. Observe, for example, that $(-x)^4 = x^4$ and $(-x)^{-3} = -x^{-3}$.

Since $(-x)^2 = x^2$, any function that depends only on $x^2$ is even. For instance, the absolute value function $y = |x| = \sqrt{x^2}$ is even.

The graph of an even function is *symmetric about the y-axis.* A horizontal straight line drawn from a point on the graph to the $y$-axis will, if continued an equal distance on the other side of the $y$-axis, come to another point on the graph. (See Figure P.50(a).)

The graph of an odd function is *symmetric about the origin.* A straight line drawn from a point on the graph to the origin will, if continued an equal distance on the other side of the origin, come to another point on the graph. If an odd function $f$ is defined at $x = 0$, then its value must be zero there: $f(0) = 0$. (See Figure P.50(b).)

If $f(x)$ is even (or odd), then so is any constant multiple of $f(x)$ such as $2f(x)$ or $-5f(x)$. Sums (and differences) of even functions are even; sums (and differences) of odd functions are odd. For example, $f(x) = 3x^4 - 5x^2 - 1$ is even, since it is the sum of three even functions: $3x^4$, $-5x^2$, and $-1 = -x^0$. Similarly, $4x^3 - (2/x)$ is an odd function. The function $g(x) = x^2 - 2x$ is the sum of an even function and an odd function and is itself neither even nor odd.

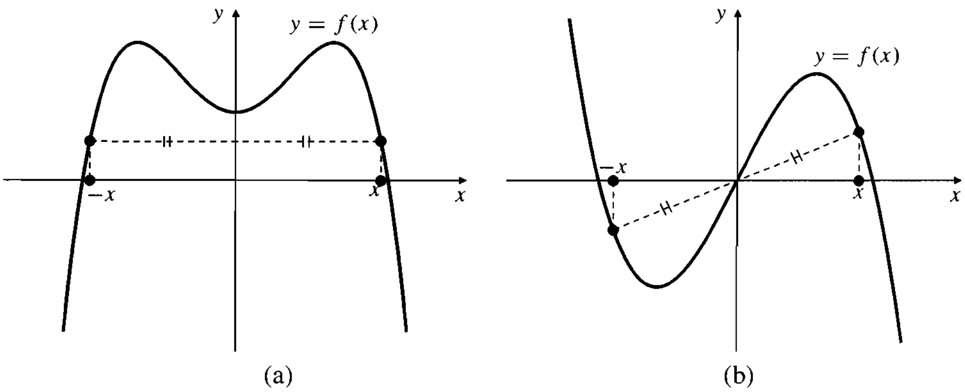



**Figure P.50**

(a) The graph of an even function is symmetric about the $y$-axis

(b) The graph of an odd function is symmetric about the origin

Other kinds of symmetry are also possible. For example, the function $g(x) = x^2 - 2x$ can be written in the form $g(x) = (x-1)^2 - 1$. This shows that the values of $g(1 \pm u)$ are equal, so the graph (Figure P.51(a)) is symmetric about the vertical line $x = 1$; it is the parabola $y = x^2$ shifted 1 unit to the right and 1 unit down. Similarly, the graph of $h(x) = x^3 + 1$ is symmetric about the point $(0, 1)$ (Figure P.51(b)).

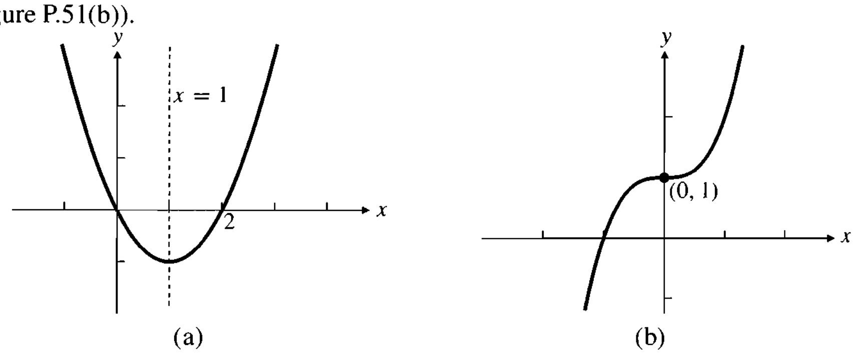



**Figure P.51**

(a) The graph of $g(x) = x^2 - 2x$ is symmetric about $x = 1$

(b) The graph of $y = h(x) = x^3 + 1$ is symmetric about $(0, 1)$

## Reflections in Straight Lines

The image of an object reflected in a plane mirror appears to be as far behind the mirror as the object is in front of it. Thus, the mirror bisects at right angles the line from a point in the object to the corresponding point in the image. Given a line $L$ and a point $P$ not on $L$, we call a point $Q$ the **reflection**, or the **mirror image**, of $P$ in $L$ if $L$ is the right bisector of the line segment $PQ$. The reflection of any graph $G$ in $L$ is the graph consisting of the reflections of all the points of $G$.

Certain reflections of graphs are easily described in terms of the equations of the graphs:

**Reflections in special lines**

1. Substituting $-x$ in place of $x$ in an equation in $x$ and $y$ corresponds to reflecting the graph of the equation in the $y$-axis.
2. Substituting $-y$ in place of $y$ in an equation in $x$ and $y$ corresponds to reflecting the graph of the equation in the $x$-axis.
3. Substituting $a - x$ in place of $x$ in an equation in $x$ and $y$ corresponds to reflecting the graph of the equation in the line $x = a/2$.
4. Substituting $b - y$ in place of $y$ in an equation in $x$ and $y$ corresponds to reflecting the graph of the equation in the line $y = b/2$.
5. Interchanging $x$ and $y$ in an equation in $x$ and $y$ corresponds to reflecting the graph of the equation in the line $y = x$.

**EXAMPLE 9** Describe and sketch the graph of $y = \sqrt{2-x} - 3$.

***Solution*** The graph of $y = \sqrt{2-x}$ is the reflection of the graph of $y = \sqrt{x}$

(Figure P.40) in the vertical line $x = 1$. The graph of $y = \sqrt{2-x} - 3$ is the result of lowering this reflection by 3 units. See Figure P.52(a).

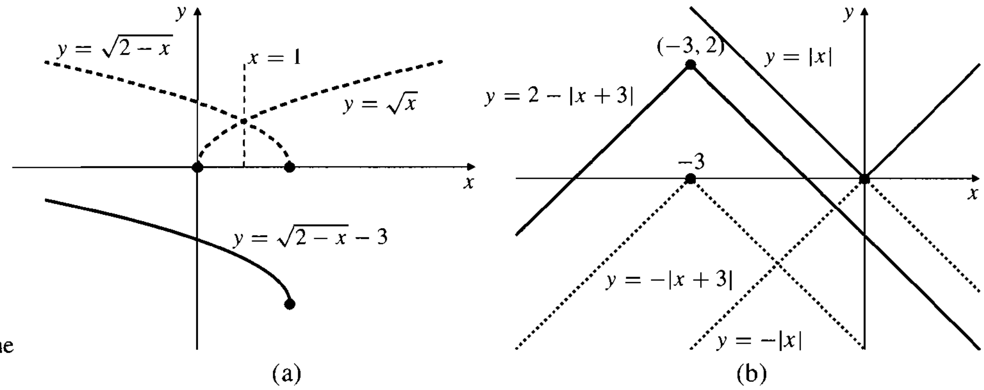



**Figure P.52**

(a) Constructing the graph of $y = \sqrt{2-x} - 3$

(b) Transforming $y = |x|$ to produce the coloured graph

**EXAMPLE 10** Express the equation of the coloured graph in Figure P.52(b) in terms of the absolute value function $|x|$.

***Solution*** We can get the coloured graph by first reflecting the graph of $|x|$ (Figure P.46) in the $x$-axis and then shifting the reflection left 3 units and up 2 units. The reflection of $y = |x|$ in the $x$-axis has equation $-y = |x|$, or $y = -|x|$. Shifting this left 3 units gives $y = -|x+3|$. Finally, shifting up 2 units gives $y = 2 - |x+3|$, which is the desired equation.

## Defining and Graphing Functions with Maple

Many of the calculations and graphs encountered in studying calculus can be produced using a computer algebra system such as Maple or Mathematica. Here and there, throughout this book, we will include examples illustrating how to get Maple to perform such tasks. (The examples were done with Maple 10, but most of them will work with earlier or later versions of Maple as well.)

We begin with an example showing how to define a function in Maple and then plot its graph. We show in colour the input you type into Maple and in black Maple's response. Let us define the function $f(x) = x^3 - 2x^2 - 12x + 1$.

```
> f := x -> x^3-2*x^2-12*x+1; <enter>
```

$$f := x \longrightarrow x^3 - 2x^2 - 12x + 1$$

Note the use of `:=` to indicate the symbol to the left is being defined and the use of `->` to indicate the rule for the construction of $f(x)$ from $x$. Also note that Maple uses the asterisk `*` to indicate multiplication and the caret `^` to indicate an exponent. A Maple instruction should end with a semicolon `;` (or a colon `:` if no output is desired) before the Enter key is pressed. Hereafter we will not show the `<enter>` in our input.

We can now use $f$ as an ordinary function:

```
> f(t)+f(1);
```

$$t^3 - 2t^2 - 12t - 11$$

The following command results in a plot of the graph of $f$ on the interval $[-4, 5]$ shaon in Figure P.53.

```
> plot(f(x), x=-4..5);
```

We could have specified the expression `x^3-2*x^2-12*x+1` directly in the plot command instead of first defining the function $f(x)$. Note the use of two dots `..` to separate the left and right endpoints of the plot interval. Other options can be included in the plot command; all such options are separated with commas. You can specify the range of values of $y$ in addition to that for $x$ (which is required), and you can specify `scaling=CONSTRAINED` if you want equal unit distances on both axes. (This would be a bad idea for the graph of our $f(x)$. Why?)

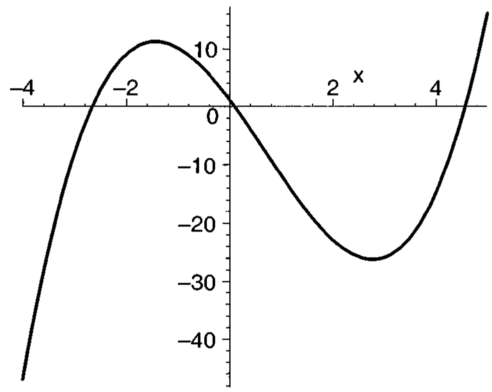



**Figure P.53** A Maple plot

When using a graphing calculator or computer graphing software things can go horribly wrong in some circumstances. The following example illustrates the catastrophic effects that **round-off error** can have

**A Numerical Monster**

**EXAMPLE 11** Consider the function $g(x) = \dfrac{|1+x|-1}{x}$.

If $x > -1$, then $|1+x| = 1+x$, so the formula for $g(x)$ simplifies to $g(x) = \dfrac{(1+x)-1}{x} = \dfrac{x}{x} = 1$, at least provided $x \neq 0$. Thus the graph of $g$ on an interval lying to the right of $x = -1$ should be the horizontal line $y = 1$, possibly with a hole in it at $x = 0$. The Maple commands

```
> g := x -> (abs(1+x)-1)/x:  plot(g(x), x=-0.5..0.5);
```

lead, as expected, to the graph in Figure P.54. But plotting the same function on a very tiny interval near $x = 0$ leads to quite a different graph. The command

```
> plot([g(x),1],x=-7*10^(-16)..5*10^(-16),
    style=[point,line],numpoints=4000);
```

produces the graph in Figure P.55.

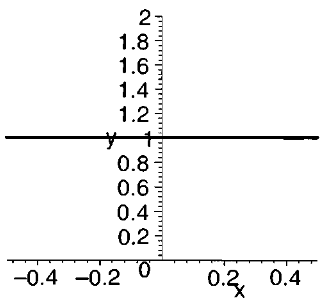



**Figure P.54** The graph of $y = g(x)$ on the interval $[-0.5, 0.5]$

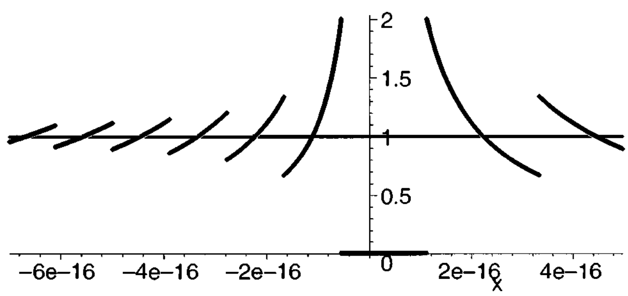



**Figure P.55** The graphs of $y = g(x)$ (colour) and $y = 1$ (black) on the interval $[-7 \times 10^{-16}, 5 \times 10^{-16}]$

The coloured arcs and short line through the origin are the graph of $y = g(x)$ plotted as 4,000 individual points over the interval from $-7 \times 10^{-16}$ to $5 \times 10^{-16}$. For comparison sake, the black horizontal line $y = 1$ is also plotted. What makes the graph of $g$ so strange on this interval is the fact that Maple can only represent finitely many real numbers in its finite memory. If the number $x$ is too close to zero, Maple cannot tell the difference between $1 + x$ and 1, so it calculates $1 - 1 = 0$ for the numerator, and uses $g(x) = 0$ in the plot. This seems to happen between about $-0.5 \times 10^{-16}$ and $0.8 \times 10^{-16}$ (the coloured horizontal line). As we move further away from the origin, Maple can tell the difference between $1 + x$ and 1, but loses most of the significant figures in the representation of $x$ when it adds 1, and these remain lost when it subtracts 1 again. Thus the numerator remains constant over short intervals while the denominator increases as $x$ moves away from 0. In those intervals the fraction behaves like $constant/x$ so the arcs are hyperbolas, sloping downward away from the origin. The effect diminishes the farther $x$ moves away from 0, as more of its significant figures are retained by Maple. It should be noted that the reason we used the absolute value of $1 + x$ instead of just $1 + x$ is that this forced Maple to add the $x$ to the 1 before subtracting the second 1. (If we had used $(1 + x) - 1$ as the numerator for $g(x)$, Maple would have simplified it algebraically and obtained $g(x) = 1$ before using any values of $x$ for plotting.)

In later chapters we will encounter more such strange behaviour (which we call **numerical monsters**) in the context of calculator and computer calculations with floating point (i.e. real) numbers. They are a necessary consequence of the limitations of such hardware and software, and are not restricted to Maple, though they may show up somewhat differently with other software. It is necessary to be aware of how calculators and computers do arithmetic in order to be able to use them effectively without falling into errors that you do not recognize as such.

One final comment about Figure P.55: the graph of $y = g(x)$ was plotted as individual points, rather than a line as was $y = 1$, in order to make the jumps between consecutive arcs more obvious. Had we omitted the `style=[point,line]` option in the plot command, the default line style would have been used for both graphs and the arcs in the graph of $g$ would have been connected with vertical line segments. Note how the command called for the plotting of two different functions by listing them within square brackets, and how the corresponding styles were correspondingly listed.

## EXERCISES P.4

In Exercises 1–6, find the domain and range of each function.

**1.** $f(x) = 1 + x^2$

**2.** $f(x) = 1 - \sqrt{x}$

**3.** $G(x) = \sqrt{8 - 2x}$

**4.** $F(x) = 1/(x - 1)$

**5.** $h(t) = \dfrac{t}{\sqrt{2 - t}}$

**6.** $g(x) = \dfrac{1}{1 - \sqrt{x - 2}}$

**7.** Which of the graphs in Figure P.56 are graphs of functions $y = f(x)$? Why?

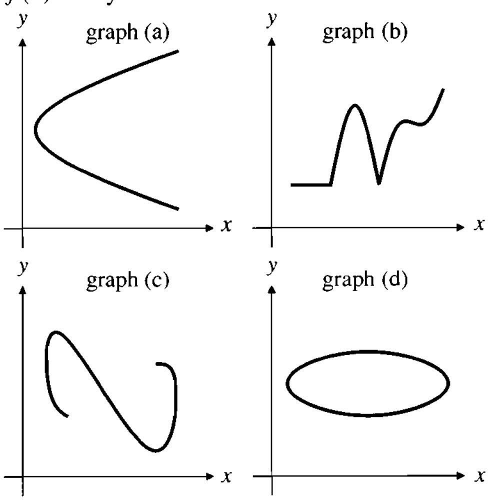



**Figure P.56**

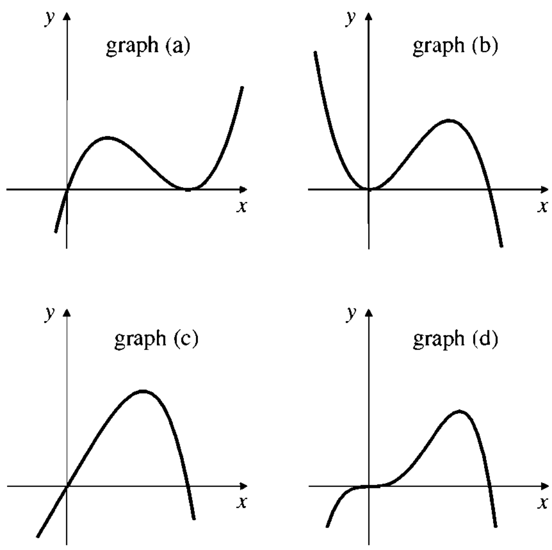



**Figure P.57**

**8.** Figure P.57 shows the graphs of the functions: (i) $x - x^4$, (ii) $x^3 - x^4$, (iii) $x(1 - x)^2$, (iv) $x^2 - x^3$. Which graph corresponds to which function?

In Exercises 9–10, sketch the graph of the function $f$ by first making a table of values of $f(x)$ at $x = 0$, $x = \pm 1/2$, $x = \pm 1$, $x = \pm 3/2$, and $x = \pm 2$.

**9.** $f(x) = x^4$

**10.** $f(x) = x^{2/3}$

In Exercises 11–22, what (if any) symmetry does the graph of $f$ possess? In particular, is $f$ even or odd?

**11.** $f(x) = x^2 + 1$

**12.** $f(x) = x^3 + x$

**13.** $f(x) = \dfrac{x}{x^2 - 1}$

**14.** $f(x) = \dfrac{1}{x^2 - 1}$

**15.** $f(x) = \dfrac{1}{x - 2}$

**16.** $f(x) = \dfrac{1}{x + 4}$

**17.** $f(x) = x^2 - 6x$

**18.** $f(x) = x^3 - 2$

**19.** $f(x) = |x^3|$

**20.** $f(x) = |x + 1|$

**21.** $f(x) = \sqrt{2x}$

**22.** $f(x) = \sqrt{(x - 1)^2}$

Sketch the graphs of the functions in Exercises 23–38.

**23.** $f(x) = -x^2$

**24.** $f(x) = 1 - x^2$

**25.** $f(x) = (x - 1)^2$

**26.** $f(x) = (x - 1)^2 + 1$

**27.** $f(x) = 1 - x^3$

**28.** $f(x) = (x + 2)^3$

**29.** $f(x) = \sqrt{x} + 1$

**30.** $f(x) = \sqrt{x + 1}$

**31.** $f(x) = -|x|$

**32.** $f(x) = |x| - 1$

**33.** $f(x) = |x - 2|$

**34.** $f(x) = 1 + |x - 2|$

**35.** $f(x) = \dfrac{2}{x + 2}$

**36.** $f(x) = \dfrac{1}{2 - x}$

**37.** $f(x) = \dfrac{x}{x + 1}$

**38.** $f(x) = \dfrac{x}{1 - x}$

In Exercises 39–46, $f$ refers to the function with domain $[0, 2]$ and range $[0, 1]$, whose graph is shown in Figure P.58. Sketch the graphs of the indicated functions and specify their domains and ranges.

**39.** $f(x) + 2$

**40.** $f(x) - 1$

**41.** $f(x + 2)$

**42.** $f(x - 1)$

**43.** $-f(x)$

**44.** $f(-x)$

**45.** $f(4 - x)$

**46.** $1 - f(1 - x)$

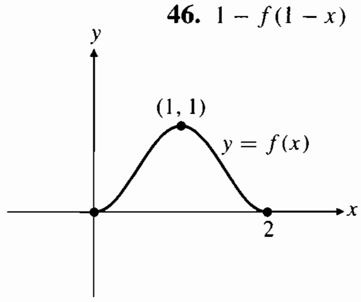



**Figure P.58**

It is often quite difficult to determine the range of a function exactly. In Exercises 47–48, use a graphing utility (calculator or computer) to graph the function $f$, and by zooming in on the graph determine the range of $f$ with accuracy of 2 decimal places.

**47.** $f(x) = \dfrac{x + 2}{x^2 + 2x + 3}$

**48.** $f(x) = \dfrac{x - 1}{x^2 + x}$

In Exercises 49–52, use a graphing utility to plot the graph of the given function. Examine the graph (zooming in or out as necessary) for symmetries. About what lines and/or points are the graphs symmetric? Try to verify your conclusions algebraically.

**49.** $f(x) = x^4 - 6x^3 + 9x^2 - 1$

**50.** $f(x) = \dfrac{3 - 2x + x^2}{2 - 2x + x^2}$

**51.** $f(x) = \dfrac{x - 1}{x - 2}$

**52.** $f(x) = \dfrac{2x^2 + 3x}{x^2 + 4x + 5}$

**53.** What function $f(x)$, defined on the real line $\mathbb{R}$, is both even and odd?

# P.5 Combining Functions to Make New Functions

Functions can be combined in a variety of ways to produce new functions.

We begin by examining algebraic means of combining functions, that is, addition, subtraction, multiplication, and division..

## Sums, Differences, Products, Quotients, and Multiples

Like numbers, functions can be added, subtracted, multiplied, and divided (except where the denominator is zero) to produce new functions.

**DEFINITION 3**

If $f$ and $g$ are functions, then for every $x$ that belongs to the domains of both $f$ and $g$ we define functions $f + g$, $f - g$, $fg$, and $f/g$ by the formulas:

$$(f + g)(x) = f(x) + g(x)$$
$$(f - g)(x) = f(x) - g(x)$$
$$(fg)(x) = f(x)g(x)$$
$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \quad \text{where } g(x) \neq 0.$$

A special case of the rule for multiplying functions shows how functions can be multiplied by constants. If $c$ is a real number, then the function $cf$ is defined for all $x$ in the domain of $f$ by

$$(cf)(x) = c\,f(x).$$

**EXAMPLE 1** Figure P.59(a) shows the graphs of $f(x) = x^2$, $g(x) = x - 1$, and their sum $(f+g)(x) = x^2 + x - 1$. Observe that the height of the graph of $f+g$ at any point $x$ is the sum of the heights of the graphs of $f$ and $g$ at that point.

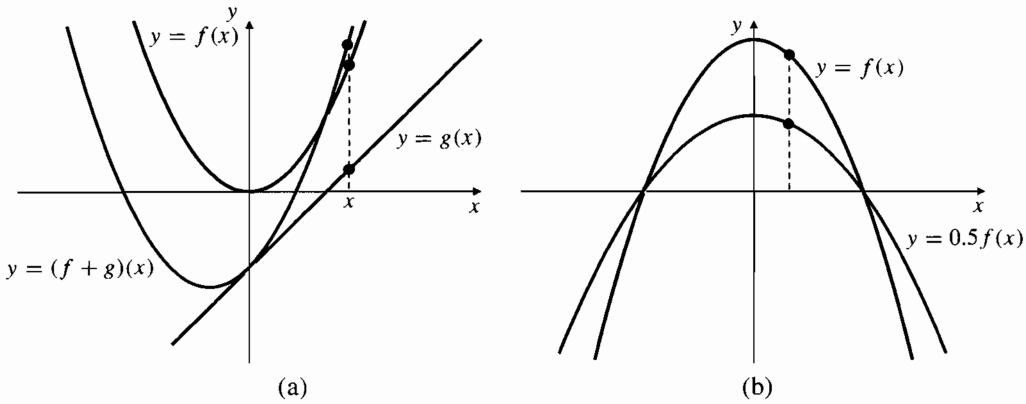



**Figure P.59**
(a) $(f+g)(x) = f(x) + g(x)$
(b) $g(x) = (0.5)f(x)$

**EXAMPLE 2** Figure P.59(b) shows the graphs of $f(x) = 2 - x^2$ and the multiple $g(x) = (0.5)f(x)$. Note how the height of the graph of $g$ at any point $x$ is half the height of the graph of $f$ there.

**EXAMPLE 3** The functions $f$ and $g$ are defined by the formulas

$$f(x) = \sqrt{x} \qquad \text{and} \qquad g(x) = \sqrt{1-x}.$$

Find formulas for the values of $3f$, $f+g$, $f-g$, $fg$, $f/g$, and $g/f$ at $x$, and specify the domains of each of these functions.

***Solution*** The information is collected in Table 2:

**Table 2.** Combinations of $f$ and $g$ and their domains

| Function | Formula | Domain |
|---|---|---|
| $f$ | $f(x) = \sqrt{x}$ | $[0, \infty)$ |
| $g$ | $g(x) = \sqrt{1-x}$ | $(-\infty, 1]$ |
| $3f$ | $(3f)(x) = 3\sqrt{x}$ | $[0, \infty)$ |
| $f+g$ | $(f+g)(x) = f(x) + g(x) = \sqrt{x} + \sqrt{1-x}$ | $[0, 1]$ |
| $f-g$ | $(f-g)(x) = f(x) - g(x) = \sqrt{x} - \sqrt{1-x}$ | $[0, 1]$ |
| $fg$ | $(fg)(x) = f(x)g(x) = \sqrt{x(1-x)}$ | $[0, 1]$ |
| $f/g$ | $\dfrac{f}{g}(x) = \dfrac{f(x)}{g(x)} = \sqrt{\dfrac{x}{1-x}}$ | $[0, 1)$ |
| $g/f$ | $\dfrac{g}{f}(x) = \dfrac{g(x)}{f(x)} = \sqrt{\dfrac{1-x}{x}}$ | $(0, 1]$ |

Note that most of the combinations of $f$ and $g$ have domains

$$[0, \infty) \cap (-\infty, 1] = [0, 1],$$

the intersection of the domains of $f$ and $g$. However, the domains of the two quotients $f/g$ and $g/f$ had to be restricted further to remove points where the denominator was zero.

## Composite Functions

There is another method, called **composition**, by which two functions can be combined to form a new function.

**DEFINITION 4**

**Composite functions**

If $f$ and $g$ are two functions, the **composite** function $f \circ g$ is defined by

$$f \circ g(x) = f(g(x)).$$

The domain of $f \circ g$ consists of those numbers $x$ in the domain of $g$ for which $g(x)$ is in the domain of $f$. In particular, if the range of $g$ is contained in the domain of $f$, then the domain of $f \circ g$ is just the domain of $g$.

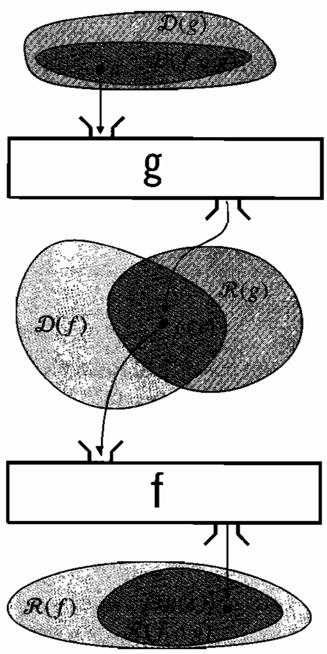


**Figure P.60** $f \circ g(x) = f(g(x))$

As shown in Figure P.60, forming $f \circ g$ is equivalent to arranging "function machines" $g$ and $f$ in an "assembly line" so that the output of $g$ becomes the input of $f$.

In calculating $f \circ g(x) = f(g(x))$ we first calculate $g(x)$ and then calculate $f$ of the result. We call $g$ the *inner* function and $f$ the *outer* function of the composition. We can, of course, also calculate the composition $g \circ f(x) = g(f(x))$, where $f$ is the inner function, the one that gets calculated first, and $g$ is the outer function, which gets calculated last. The functions $f \circ g$ and $g \circ f$ are usually quite different, as the following example shows.

**EXAMPLE 4** Given $f(x) = \sqrt{x}$ and $g(x) = x + 1$, calculate the four composite functions $f \circ g(x)$, $g \circ f(x)$, $f \circ f(x)$, and $g \circ g(x)$, and specify the domain of each.

***Solution*** Again, we collect the results in a table

**Table 3.** Composites of $f$ and $g$ and their domains

| Function | Formula | Domain |
|---|---|---|
| $f$ | $f(x) = \sqrt{x}$ | $[0, \infty)$ |
| $g$ | $g(x) = x + 1$ | $\mathbb{R}$ |
| $f \circ g$ | $f \circ g(x) = f(g(x)) = f(x+1) = \sqrt{x+1}$ | $[-1, \infty)$ |
| $g \circ f$ | $g \circ f(x) = g(f(x)) = g(\sqrt{x}) = \sqrt{x} + 1$ | $[0, \infty)$ |
| $f \circ f$ | $f \circ f(x) = f(f(x)) = f(\sqrt{x}) = \sqrt{\sqrt{x}} = x^{1/4}$ | $[0, \infty)$ |
| $g \circ g$ | $g \circ g(x) = g(g(x)) = g(x+1) = (x+1) + 1 = x + 2$ | $\mathbb{R}$ |

To see why, for example, the domain of $f \circ g$ is $[-1, \infty)$, observe that $g(x) = x + 1$ is defined for all real $x$ but belongs to the domain of $f$ only if $x + 1 \geq 0$, that is, if $x \geq -1$.

**EXAMPLE 5** If $G(x) = \dfrac{1-x}{1+x}$, calculate $G \circ G(x)$ and specify its domain.

***Solution*** We calculate

$$G \circ G(x) = G(G(x)) = G\left(\frac{1-x}{1+x}\right) = \frac{1 - \dfrac{1-x}{1+x}}{1 + \dfrac{1-x}{1+x}} = \frac{1 + x - 1 + x}{1 + x + 1 - x} = x.$$

Because the resulting function, $x$, is defined for all real $x$, we might be tempted to say that the domain of $G \circ G$ is $\mathbb{R}$. This is wrong! To belong to the domain of $G \circ G$, $x$ must satisfy two conditions:

(i) $x$ must belong to the domain of $G$, and

(ii) $G(x)$ must belong to the domain of $G$.

The domain of $G$ consists of all real numbers *except* $x = -1$. If we exclude $x = -1$ from the domain of $G \circ G$, condition (i) will be satisfied. Now observe that the equation $G(x) = -1$ has no solution $x$, since it is equivalent to $1 - x = -(1 + x)$ or $1 = -1$. Therefore, all numbers $G(x)$ belong to the domain of $G$, and condition (ii) is satisfied with no further restrictions on $x$. The domain of $G \circ G$ is $(-\infty, -1) \cup (-1, \infty)$, that is, all real numbers except $-1$.

## Piecewise Defined Functions

Sometimes it is necessary to define a function by using different formulas on different parts of its domain. One example is the absolute value function

$$|x| = \begin{cases} x & \text{if } x \ge 0 \\ -x & \text{if } x < 0. \end{cases}$$

Here are some other examples. (Note how we use solid and hollow dots in their graphs to indicate, respectively, which endpoints do or do not lie on various parts of the graph.

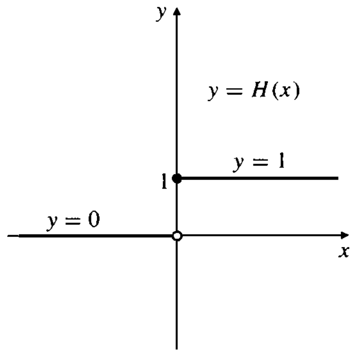



**Figure P.61** The Heaviside function

**EXAMPLE 6** **The Heaviside function.** The Heaviside function (or unit step function) (Figure P.61) is defined by

$$H(x) = \begin{cases} 1 & \text{if } x \ge 0 \\ 0 & \text{if } x < 0. \end{cases}$$

The function $H(t)$ can be used, for example, to model the voltage applied to an electric circuit by a one volt battery if a switch in the circuit is closed at time $t = 0$.

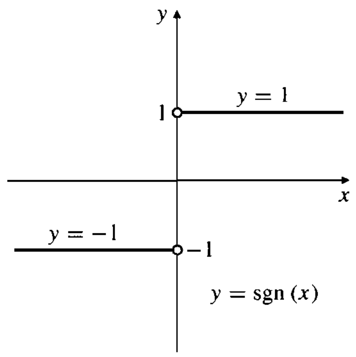



**Figure P.62** The signum function

**EXAMPLE 7** **The signum function.** The signum function (Figure P.62) is defined as follows:

$$\operatorname{sgn}(x) = \frac{x}{|x|} = \begin{cases} 1 & \text{if } x > 0, \\ -1 & \text{if } x < 0, \\ \text{undefined} & \text{if } x = 0. \end{cases}$$

The name *signum* is the Latin word meaning "sign." The value of the sgn$(x)$ tells whether $x$ is positive or negative. Since 0 is neither positive nor negative, sgn (0) is not defined. The signum function is an odd function.

**EXAMPLE 8** The function

$$f(x) = \begin{cases} (x+1)^2 & \text{if } x < -1, \\ -x & \text{if } -1 \le x < 1, \\ \sqrt{x-1} & \text{if } x \ge 1, \end{cases}$$

is defined on the whole real line but has values given by three different formulas depending on the position of $x$. Its graph is shown in Figure P.63(a).

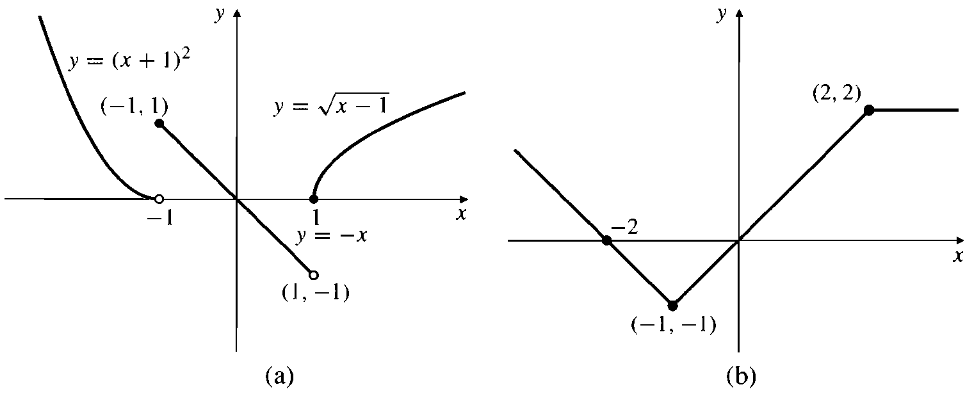



**Figure P.63** Piecewise defined functions

**EXAMPLE 9** Find a formula for function $g(x)$ graphed in Figure P.63(b).

***Solution*** The graph consists of parts of three lines. For the part $x < -1$, the line has slope $-1$ and $x$-intercept $-2$, so its equation is $y = -(x+2)$. The middle section is the line $y = x$ for $-1 \le x \le 2$. The right section is $y = 2$ for $x > 2$. Combining these formulas, we write

$$g(x) = \begin{cases} -(x+2) & \text{if } x < -1 \\ x & \text{if } -1 \le x \le 2 \\ 2 & \text{if } x > 2. \end{cases}$$

Unlike the previous example, it does not matter here which of the two possible formulas we use to define $g(-1)$, since both give the same value. The same is true for $g(2)$.

The following two functions could be defined by different formulas on every interval between consecutive integers, but we will use an easier way to define them.

**EXAMPLE 10** **The greatest integer function.** The function whose value at any number $x$ is the *greatest integer less than or equal to* $x$ is called the **greatest integer function**, or the **integer floor function**. It is denoted $\lfloor x \rfloor$, or, in some books, $[x]$ or $[[x]]$. The graph of $y = \lfloor x \rfloor$ is given in Figure P.64(a). Observe that

$$\begin{array}{llll} \lfloor 2.4 \rfloor = 2, & \lfloor 1.9 \rfloor = 1, & \lfloor 0 \rfloor = 0, & \lfloor -1.2 \rfloor = -2, \\ \lfloor 2 \rfloor = 2, & \lfloor 0.2 \rfloor = 0, & \lfloor -0.3 \rfloor = -1, & \lfloor -2 \rfloor = -2. \end{array}$$

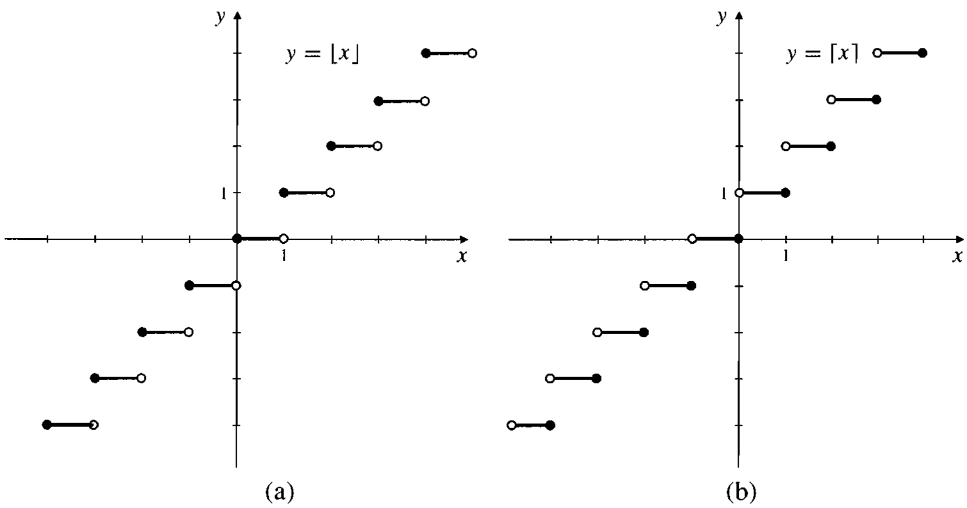



**Figure P.64**
(a) The greatest integer function $\lfloor x \rfloor$
(b) The least integer function $\lceil x \rceil$

**EXAMPLE 11** **The least integer function.** The function whose value at any number $x$ is the *smallest integer greater than or equal to* $x$ is called the **least integer function**, or the **integer ceiling function**. It is denoted $\lceil x \rceil$. Its graph is given in Figure P.64(b). For positive values of $x$, this function might represent, for example, the cost of parking $x$ hours in a parking lot that charges \$1 for each hour or part of an hour.

## EXERCISES P.5

In Exercises 1–2, find the domains of the functions $f+g$, $f-g$, $fg$, $f/g$, and $g/f$, and give formulas for their values.

**1.** $f(x)=x, \quad g(x)=\sqrt{x-1}$

**2.** $f(x)=\sqrt{1-x}, \quad g(x)=\sqrt{1+x}$

Sketch the graphs of the functions in Exercises 3–6 by combining the graphs of simpler functions from which they are built up.

**3.** $x-x^2$ **4.** $x^3-x$

**5.** $x+|x|$ **6.** $|x|+|x-2|$

**7.** If $f(x)=x+5$ and $g(x)=x^2-3$, find the following:
(a) $f\circ g(0)$ (b) $g(f(0))$
(c) $f(g(x))$ (d) $g\circ f(x)$
(e) $f\circ f(-5)$ (f) $g(g(2))$
(g) $f(f(x))$ (h) $g\circ g(x)$

In Exercises 8–10, construct the following composite functions and specify the domain of each.
(a) $f\circ f(x)$ (b) $f\circ g(x)$
(c) $g\circ f(x)$ (d) $g\circ g(x)$

**8.** $f(x)=2/x, \quad g(x)=x/(1-x)$

**9.** $f(x)=1/(1-x), \quad g(x)=\sqrt{x-1}$

**10.** $f(x)=(x+1)/(x-1), \quad g(x)=\operatorname{sgn}(x)$

Find the missing entries in Table 4 (Exercises 11–16).

**Table 4.**

| | $f(x)$ | $g(x)$ | $f\circ g(x)$ |
|---|---|---|---|
| **11.** | $x^2$ | $x+1$ | |
| **12.** | | $x+4$ | $x$ |
| **13.** | $\sqrt{x}$ | | $\|x\|$ |
| **14.** | | $x^{1/3}$ | $2x+3$ |
| **15.** | $(x+1)/x$ | | $x$ |
| **16.** | | $x-1$ | $1/x^2$ |

**17.** Use a graphing utility to examine in order the graphs of the functions

$$y=\sqrt{x}, \qquad y=2+\sqrt{x},$$
$$y=2+\sqrt{3+x}, \qquad y=1/(2+\sqrt{3+x}).$$

Describe the effect on the graph of the change made in the function at each stage.

**18.** Repeat the previous exercise for the functions

$$y=2x, \qquad y=2x-1, \qquad y=1-2x,$$
$$y=\sqrt{1-2x}, \qquad y=\frac{1}{\sqrt{1-2x}}, \qquad y=\frac{1}{\sqrt{1-2x}}-1.$$

In Exercises 19–24, $f$ refers to the function with domain [0, 2] and range [0, 1], whose graph is shown in Figure P.65. Sketch the graphs of the indicated functions, and specify their domains and ranges.

**19.** $2f(x)$ **20.** $-(1/2)f(x)$

**21.** $f(2x)$ **22.** $f(x/3)$

**23.** $1+f(-x/2)$ **24.** $2f((x-1)/2)$

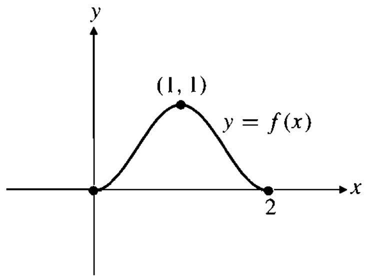



Figure P.65

In Exercises 25–26, sketch the graphs of the given functions.

**25.** $f(x)=\begin{cases} x & \text{if } 0\le x\le 1 \\ 2-x & \text{if } 1<x\le 2\end{cases}$

**26.** $g(x)=\begin{cases} \sqrt{x} & \text{if } 0\le x\le 1 \\ 2-x & \text{if } 1<x\le 2\end{cases}$

**27.** Find all real values of the constants $A$ and $B$ for which the function $F(x)=Ax+B$ satisfies:
(a) $F\circ F(x)=F(x)$ for all $x$.
(b) $F\circ F(x)=x$ for all $x$.

### Greatest and least integer functions

**28.** For what values of $x$ is (a) $\lfloor x\rfloor=0$? (b) $\lceil x\rceil=0$?

**29.** What real numbers $x$ satisfy the equation $\lfloor x\rfloor=\lceil x\rceil$?

**30.** True or false: $\lceil -x\rceil=-\lfloor x\rfloor$ for all real $x$?

**31.** Sketch the graph of $y=x-\lfloor x\rfloor$.

**32.** Sketch the graph of the function

$$f(x)=\begin{cases}\lfloor x\rfloor & \text{if } x\ge 0\\ \lceil x\rceil & \text{if } x<0.\end{cases}$$

Why is $f(x)$ called *the integer part of* $x$?

**Even and odd functions**

**33.** Assume that $f$ is an even function, $g$ is an odd function, and both $f$ and $g$ are defined on the whole real line $\mathbb{R}$. Is each of the following functions even, odd, or neither?

$$f+g, \quad fg, \quad f/g, \quad g/f, \quad f^2=ff, \quad g^2=gg$$

$$f\circ g, \quad g\circ f, \quad f\circ f, \quad g\circ g$$

**34.** If $f$ is both an even and an odd function, show that $f(x)=0$ at every point of its domain.

**35.** Let $f$ be a function whose domain is symmetric about the origin, that is, $-x$ belongs to the domain whenever $x$ does.

(a) Show that $f$ is the sum of an even function and an odd function:

$$f(x)=E(x)+O(x),$$

where $E$ is an even function and $O$ is an odd function. *Hint:* Let $E(x)=(f(x)+f(-x))/2$. Show that $E(-x)=E(x)$, so that $E$ is even. Then show that $O(x)=f(x)-E(x)$ is odd.

(b) Show that there is only one way to write $f$ as the sum of an even and an odd function. *Hint:* One way is given in part (a). If also $f(x)=E_1(x)+O_1(x)$, where $E_1$ is even and $O_1$ is odd, show that $E-E_1=O_1-O$ and then use Exercise 34 to show that $E=E_1$ and $O=O_1$.

# P.6 Polynomials and Rational Functions

Among the easiest functions to deal with in calculus are polynomials. These are sums of terms each of which is a constant multiple of a nonnegative integer power of the variable of the function:

**DEFINITION 5**

A **polynomial** is a function $P$ whose value at $x$ is

$$P(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_2x^2+a_1x+a_0,$$

where $a_n, a_{n-1}, \ldots, a_2, a_1$, and $a_0$, called the **coefficients** of the polymonial, are constants and, if $n>0$, then $a_n\neq 0$. The number $n$, the degree of the highest power of $x$ in the polynomial, is called the **degree** of the polynomial. (The degree of the zero polynomial is not defined.)

For example,

| | |
|---|---|
| $3$ | is a polynomial of degree 0. |
| $2-x$ | is a polynomial of degree 1. |
| $2x^3-17x+1$ | is a polynomial of degree 3. |

Generally, we assume that the polynomials we deal with are *real polynomials*, that is, their coefficients are real numbers rather than more general complex numbers; often the coefficients will be integers or rational numbers. Polynomials play a role in the study of functions somewhat analogous to the role played by integers in the study of numbers. For instance, just as we always get an integer result if we add, subtract, or multiply two integers, we always get a polynomial result if we add, subtract, or multiply two polynomials. Adding or subtracting polynomials produces a polynomial whose degree does not exceed the larger of the two degrees of the polynomials being combined. Multiplying two polynomials of degrees $m$ and $n$ produces a product polynomial of degree $m+n$. For instance, for the product

$$(x^2+1)(x^3-x-2)=x^5-2x^2-x-2,$$

the two factors have degrees 2 and 3, so the result has degree 5.

Just as the quotient of two integers is often not an integer but is called a **rational** number, the quotient of two polynomials is often not a polynomial, but is instead called **a rational function**.

$$\frac{2x^3 - 3x^2 + 3x + 4}{x^2 + 1} \qquad \text{is a rational function.}$$

When we divide a positive integer $a$ by a smaller positive integer $b$, we can obtain an integer quotient $q$ and an integer remainder $r$ satisfying $0 \leq r < b$ and hence write the fraction $a/b$ (in a unique way) as the sum of the integer $q$ and another fraction whose numerator (the remainder $r$) is smaller than its denominator $b$. For instance,

$$\frac{7}{3} = 2 + \frac{1}{3}; \qquad \text{the quotient is 2, the remainder is 1.}$$

Similarly, if $A_m$ and $B_n$ are polynomials having degrees $m$ and $n$, respectively, and if $m > n$, then we can express the rational function $A_m/B_n$ (in a unique way) as the sum of a quotient polynomial $Q_{m-n}$ of degree $m - n$ and another rational function $R_k/B_n$ where the numerator polynomial $R_k$ (the remainder in the division) is either zero or has degree $k < n$:

$$\frac{A_m(x)}{B_n(x)} = Q_{m-n}(x) + \frac{R_k(x)}{B_n(x)}. \qquad \textbf{(The Division Algorithm)}$$

We calculate the quotient and remainder polynomials by using "long division" or an equivalent method.

**EXAMPLE 1** Write the division algorithm for $\dfrac{2x^3 - 3x^2 + 3x + 4}{x^2 + 1}$.

***Solution*** **METHOD I.** Use long division:

$$\begin{array}{r|rrrr}
 & 2x & -\ 3 & & \\ \cline{2-5}
x^2 + 1 & 2x^3 & -\ 3x^2 & +\ 3x & +\ 4 \\
 & 2x^3 & & +\ 2x & \\ \cline{2-5}
 & & -3x^2 & +\ x & +\ 4 \\
 & & -3x^2 & & -\ 3 \\ \cline{3-5}
 & & & x & +\ 7
\end{array}$$

Thus,

$$\frac{2x^3 - 3x^2 + 3x + 4}{x^2 + 1} = 2x - 3 + \frac{x + 7}{x^2 + 1}.$$

The quotient is $2x - 3$, and the remainder is $x + 7$.

**METHOD II.** Use short division; add appropriate lower-degree terms to the terms of the numerator that have degrees not less than the degree of the denominator to enable factoring out the denominator, and then subtract those terms off again.

$$\begin{aligned}
& 2x^3 - 3x^2 + 3x + 4 \\
= \ & 2x^3 + 2x - 3x^2 - 3 + 3x + 4 - 2x + 3 \\
= \ & 2x(x^2 + 1) - 3(x^2 + 1) + x + 7,
\end{aligned}$$

from which it follows at once that

$$\frac{2x^3 - 3x^2 + 3x + 4}{x^2 + 1} = 2x - 3 + \frac{x + 7}{x^2 + 1}.$$

## Roots, Zeros, and Factors

A number $r$ is called a **root** or **zero** of the polynomial $P$ if $P(r) = 0$. For example, $P(x) = x^3 - 4x$ has three roots: 0, 2, and $-2$; substituting any of these numbers for $x$ makes $P(x) = 0$. In this context the terms "root" and "zero" are often used interchangeably. It is technically more correct to call a number $r$ satisfying $P(r) = 0$ a *zero* of the polynomial *function* $P$ and a *root* of the *equation* $P(x) = 0$, and later in this book we will follow this convention more closely. But for now, to avoid confusion with the *number* zero, we will prefer to use "root" rather than "zero" even when referring to the polynomial $P$ rather than the equation $P(x) = 0$.

The Fundamental Theorem of Algebra (see Appendix II) states that every polynomial of degree at least 1 has a root (although the root might be a complex number). For example, the linear (degree 1) polynomial $ax + b$ has the root $-b/a$ since $a(-b/a) + b = 0$. A constant polynomial (one of degree zero) cannot have any roots unless it is the zero polynomial, in which case every number is a root.

Real polynomials need not always have real roots; the polynomial $x^2 + 4$ is never zero for any real number $x$, but it is zero if $x$ is either of the two complex numbers $2i$ and $-2i$, where $i$ is the so-called imaginary unit satisfying $i^2 = -1$. (See Appendix I for a discussion of complex numbers.) The numbers $2i$ and $-2i$ are *complex conjugates of each other*. Any complex roots of a real polynomial must occur in conjugate pairs. (See Appendix II for a proof of this fact.)

In our study of calculus we will often find it useful to factor polynomials into products of polynomials of lower degree, especially degree 1 or 2 (linear or quadratic polynomials). The following theorem shows the connection between linear factors and roots.

**THEOREM 1** **The Factor Theorem**

The number $r$ is a root of the polynomial $P$ of degree not less than 1 if and only if $x - r$ is a factor of $P(x)$.

***PROOF*** By the division algorithm there exists a quotient polynomial $Q$ having degree one less than that of $P$ and a remainder polynomial of degree 0 (i.e., a constant $c$) such that

$$\frac{P(x)}{x-r} = Q(x) + \frac{c}{x-r}.$$

Thus $P(x) = (x - r)Q(x) + c$, and $P(r) = 0$ if and only if $c = 0$, in which case $P(x) = (x - r)Q(x)$ and $x - r$ is a factor of $P(x)$. ■

It follows from Theorem 1 and the Fundamental Theorem of Algebra that every polynomial of degree $n \geq 1$ has $n$ roots. (If $P$ has degree $n \geq 2$, then $P$ has a zero $r$ and $P(x) = (x - r)Q(x)$, where $Q$ is a polynomial of degree $n - 1 \geq 1$, which in turn has a root, etc.) Of course, the roots of a polynomial need not all be different. The 4th degree polynomial $P(x) = x^4 - 3x^3 + 3x^2 - x = x(x - 1)^3$ has four roots; one is 0 and the other three are each equal to 1. We say that the root 1 has **multiplicity** 3 because we can divide $P(x)$ by $(x - 1)^3$ and still get zero remainder.

If $P$ is a real polynomial having a complex root $r_1 = u + iv$, where $u$ and $v$ are real and $v \neq 0$, then, as asserted above, the complex conjugate of $r_1$, namely, $r_2 = u - iv$, will also be a root of $P$. (Moreover, $r_1$ and $r_2$ will have the same multiplicity.) Thus, both $x - u - iv$ and $x - u + iv$ are factors of $P(x)$, and so, therefore, is their product

$$(x - u - iv)(x - u + iv) = (x - u)^2 + v^2 = x^2 - 2ux + u^2 + v^2,$$

which is a quadratic polynomial having no real roots. It follows that every real polynomial can be factored into a product of real (possibly repeated) linear factors and real (also possibly repeated) quadratic factors having no real zeros.

**EXAMPLE 2** What is the degree of $P(x) = x^3(x^2 + 2x + 5)^2$? What are the roots of $P$ and, what is the multiplicity of each root?

***Solution*** If $P$ is expanded, the highest power of $x$ present in the expansion is $x^3(x^2)^2 = x^7$, so $P$ has degree 7. The factor $x^3 = (x - 0)^3$ indicates that 0 is a root of $P$ having multiplicity 3. The remaining four roots will be the two roots of $x^2 + 2x + 5$, each having multiplicity 2. Now

$$\begin{aligned}\left[x^2 + 2x + 5\right]^2 &= \left[(x+1)^2 + 4\right]^2 \\ &= \left[(x + 1 + 2i)(x + 1 - 2i)\right]^2.\end{aligned}$$

Hence the seven roots of $P$ are:

$$\begin{cases} 0,\ 0,\ 0 & 0 \text{ has multiplicity 3,} \\ -1-2i,\ -1-2i & -1-2i \text{ has multiplicity 2,} \\ -1+2i,\ -1+2i & -1+2i \text{ has multiplicity 2.} \end{cases}$$

## Roots and Factors of Quadratic Polynomials

There is a well-known formula for finding the roots of a quadratic polynomial.

**The Quadratic Formula**

The two solutions of the quadratic equation

$$Ax^2 + Bx + C = 0,$$

where $A$, $B$, and $C$ are constants and $A \neq 0$, are given by

$$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}.$$

To see this, just divide the equation by $A$ and complete the square for the terms in $x$:

$$\begin{aligned} x^2 + \frac{B}{A}x + \frac{C}{A} &= 0 \\ x^2 + \frac{2B}{2A}x + \frac{B^2}{4A^2} &= \frac{B^2}{4A^2} - \frac{C}{A} \\ \left(x + \frac{B}{2A}\right)^2 &= \frac{B^2 - 4AC}{4A^2} \\ x + \frac{B}{2A} &= \pm\frac{\sqrt{B^2 - 4AC}}{2A}. \end{aligned}$$

The quantity $D = B^2 - 4AC$ that appears under the square root in the quadratic formula is called the **discriminant** of the quadratic equation or polynomial. The nature of the roots of the quadratic depends on the sign of this discriminant.

(a) If $D > 0$, then $D = k^2$ for some real constant $k$, and the quadratic has two distinct roots, $(-B + k)/(2A)$ and $(-B - k)/(2A)$.

(b) If $D = 0$, then the quadratic has only the root $-B/(2A)$, and this root has multiplicity 2. (It is called a *double root.*)

(c) If $D < 0$, then $D = -k^2$ for some real constant $k$, and the quadratic has two complex conjugate roots, $(-B + ki)/(2A)$ and $(-B - ki)/(2A)$.

**EXAMPLE 3** Find the roots of these quadratic polynomials and thereby factor the polynomials into linear factors:

(a) $x^2 + x - 1$ (b) $9x^2 - 6x + 1$ (c) $2x^2 + x + 1$.

**Solution** We use the quadratic formula to solve the corresponding quadratic equations to find the roots of the three polynomials.

(a) $A = 1, \quad B = 1, \quad C = -1$

$$x = \frac{-1 \pm \sqrt{1+4}}{2} = -\frac{1}{2} \pm \frac{\sqrt{5}}{2}$$

$$x^2 + x - 1 = \left(x + \frac{1}{2} - \frac{\sqrt{5}}{2}\right)\left(x + \frac{1}{2} + \frac{\sqrt{5}}{2}\right).$$

(b) $A = 9, \quad B = -6, \quad C = 1$

$$x = \frac{6 \pm \sqrt{36 - 36}}{18} = \frac{1}{3} \quad \text{(double root)}$$

$$9x^2 - 6x + 1 = 9\left(x - \frac{1}{3}\right)^2 = (3x - 1)^2.$$

(c) $A = 2, \quad B = 1, \quad C = 1$

$$x = \frac{-1 \pm \sqrt{1-8}}{4} = -\frac{1}{4} \pm \frac{\sqrt{7}}{4} i$$

$$2x^2 + x + 1 = 2\left(x + \frac{1}{4} - \frac{\sqrt{7}}{4} i\right)\left(x + \frac{1}{4} + \frac{\sqrt{7}}{4} i\right).$$

**Remark** There exist formulas for calculating exact roots of cubic (degree 3) and quartic (degree 4) polynomials, but, unlike the quadratic formula above, they are very complicated and almost never used. Instead, calculus will provide us with very powerful and easily used tools for approximating roots of polynomials (and solutions of much more general equations) to any desired degree of accuracy.

## Miscellaneous Factorings

Some quadratic and higher degree polynomials can be (at least partially) factored by inspection. Some simple examples include:

(a) Common Factor: $ax^2 + bx = x(ax + b)$.

(b) Difference of Squares: $x^2 - a^2 = (x - a)(x + a)$.

(c) Difference of Cubes: $x^3 - a^3 = (x - a)(x^2 + ax + a^2)$.

(d) More generally, a difference of $n$th powers for any positive integer $n$:

$$x^n - a^n = (x - a)(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \cdots + a^{n-2}x + a^{n-1}).$$

Note that $x - a$ is a factor of $x^n - a^n$ for any positive integer $n$.

(e) It is also true that if $n$ is an *odd positive integer*, then $x + a$ is a factor of $x^n + a^n$. For example,

$$x^3 + a^3 = (x + a)(x^2 - ax + a^2)$$
$$x^5 + a^5 = (x + a)(x^4 - ax^3 + a^2x^2 - a^3x + a^4).$$

Finally, we mention a trial-and-error method of factoring quadratic polynomials sometimes called *trinomial factoring*. Since

$$(x + p)(x + q) = x^2 + (p + q)x + pq,$$
$$(x - p)(x - q) = x^2 - (p + q)x + pq, \qquad \text{and}$$
$$(x + p)(x - q) = x^2 + (p - q)x - pq,$$

we can sometimes spot the factors of $x^2 + Bx + C$ by looking for factors of $|C|$ for which the sum or difference is $B$. More generally, we can sometimes factor

$$Ax^2 + Bx + C = (ax + b)(cx + d)$$

by looking for factors $a$ and $c$ of $A$ and factors $b$ and $d$ of $C$ for which $ad + bc = B$. Of course, if this fails you can always resort to the quadratic formula to find the roots and, therefore, the factors, of the quadratic polynomial.

## EXAMPLE 4

$$\begin{aligned} x^2 - 5x + 6 &= (x-3)(x-2) & p &= 3, q = 2, pq = 6, p+q = 5 \\ x^2 + 7x + 6 &= (x+6)(x+1) & p &= 6, q = 1, pq = 6, p+q = 7 \\ x^2 + \ x - 6 &= (x+3)(x-2) & p &= 3, q = -2, pq = -6, p+q = 1 \\ 2x^2 + x - 10 &= (2x+5)(x-2) & a &= 2, b = 5, c = 1, d = -2 \\ & & ac &= 2, bd = -10, ad + bc = 1. \end{aligned}$$

## EXAMPLE 5

Find the roots of the following polynomials:

(a) $x^3 - x^2 - 4x + 4$, (b) $x^4 + 3x^2 - 4$, (c) $x^5 - x^4 - x^2 + x$.

***Solution*** (a) There is an obvious common factor:

$$x^3 - x^2 - 4x + 4 = (x-1)(x^2-4) = (x-1)(x-2)(x+2).$$

The roots are 1, 2, and $-2$.

(b) This is a trinomial in $x^2$ for which there is an easy factoring:

$$x^4 + 3x^2 - 4 = (x^2+4)(x^2-1) = (x+2i)(x-2i)(x+1)(x-1).$$

The roots are $1, -1, 2i$, and $-2i$.

(c) We start with some obvious factorings:

$$\begin{aligned} x^5 - x^4 - x^2 + x = x(x^4 - x^3 - x + 1) &= x(x-1)(x^3-1) \\ &= x(x-1)^2(x^2+x+1). \end{aligned}$$

Thus 0 is a root, and 1 is a double root. The remaining two roots must come from the quadratic factor $x^2 + x + 1$, which cannot be factored easily by inspection so we use the formula:

$$x = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i.$$

## EXERCISES P.6

Find the roots of the polynomials in Exercises 1–12. If a root is repeated, give its multiplicity. Also, write each polynomial as a product of linear factors.

**1.** $x^2 + 7x + 10$ **2.** $x^2 - 3x - 10$

**3.** $x^2 + 2x + 2$ **4.** $x^2 - 6x + 13$

**5.** $16x^4 - 8x^2 + 1$ **6.** $x^4 + 6x^3 + 9x^2$

**7.** $x^3 + 1$ **8.** $x^4 - 1$

**9.** $x^6 - 3x^4 + 3x^2 - 1$ **10.** $x^5 - x^4 - 16x + 16$

**11.** $x^5 + x^3 + 8x^2 + 8$ **12.** $x^9 - 4x^7 - x^6 + 4x^4$

In Exercises 13–16, express the given rational function as the sum of a polynomial and another rational function whose numerator is either zero or has smaller degree than the denominator.

**13.** $\dfrac{x^3 - 1}{x^2 - 2}$ **14.** $\dfrac{x^2}{x^2 + 5x + 3}$

**15.** $\dfrac{x^3}{x^2 + 2x + 3}$ **16.** $\dfrac{x^4 + x^2}{x^3 + x^2 + 1}$

**17.** Show that $x - 1$ is a factor of a polynomial $P$ of positive degree if and only if the sum of the coefficients of $P$ is zero.

**18.** What condition should the coefficients of a polynomial satsify to ensure that $x + 1$ is a factor of that polynomial?

**19.** The complex conjugate of a complex number $z = u + iv$ (where $u$ and $v$ are real numbers) is the complex number $\bar{z} = u - iv$. It is shown in Appendix I that the complex conjugate of a sum (or product) of complex numbers is the sum (or product) of the complex conjugates of those numbers. Use this fact to verify that if $z = u + iv$ is a complex root of a polynomial $P$ having real coefficients, then its conjugate $\bar{z}$ is also a root of $P$.

**20.** Continuing the previous exercise, show that if $z = u + iv$ (where $u$ and $v$ are real numbers) is a complex root of a polynomial $P$ with real coefficients, then $P$ must have the real quadratic factor $x^2 - 2ux + u^2 + v^2$.

**21.** Use the result of Exercise 20 to show that if $z = u + iv$ (where $u$ and $v$ are real numbers) is a complex root of a polynomial $P$ with real coefficients, then $z$ and $\bar{z}$ are roots of $P$ having the same multiplicity.

## P.7 The Trigonometric Functions

Most people first encounter the quantities $\cos t$ and $\sin t$ as ratios of sides in a right-angled triangle having $t$ as one of the acute angles. If the sides of the triangle are labelled "hyp" for hypotenuse, "adj" for the side adjacent to angle $t$, and "opp" for the side opposite angle $t$ (see Figure P.66), then

$$\cos t = \frac{\text{adj}}{\text{hyp}} \qquad \text{and} \qquad \sin t = \frac{\text{opp}}{\text{hyp}}. \qquad (*)$$

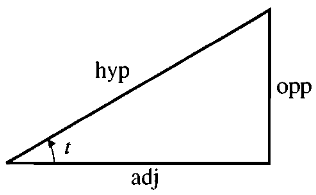



**Figure P.66** $\cos t = \text{adj}/\text{hyp}$
$\sin t = \text{opp}/\text{hyp}$

These ratios depend only on the angle $t$, not on the particular triangle, since all right-angled triangles having an acute angle $t$ are similar.

In calculus we need more general definitions of $\cos t$ and $\sin t$ as functions defined for *all real numbers* $t$, not just acute angles. Such definitions are phrased in terms of a circle rather than a triangle.

Let $C$ be the circle with centre at the origin $O$ and radius 1; its equation is $x^2 + y^2 = 1$. Let $A$ be the point $(1, 0)$ on $C$. For any real number $t$, let $P_t$ be the point on $C$ at distance $|t|$ from $A$, measured along $C$ in the counterclockwise direction if $t > 0$, and the clockwise direction if $t < 0$. For example, since $C$ has circumference $2\pi$, the point $P_{\pi/2}$ is one-quarter of the way counterclockwise around $C$ from $A$; it is the point $(0, 1)$.

We will use the arc length $t$ as a measure of the size of the angle $AOP_t$. See Figure P.67.

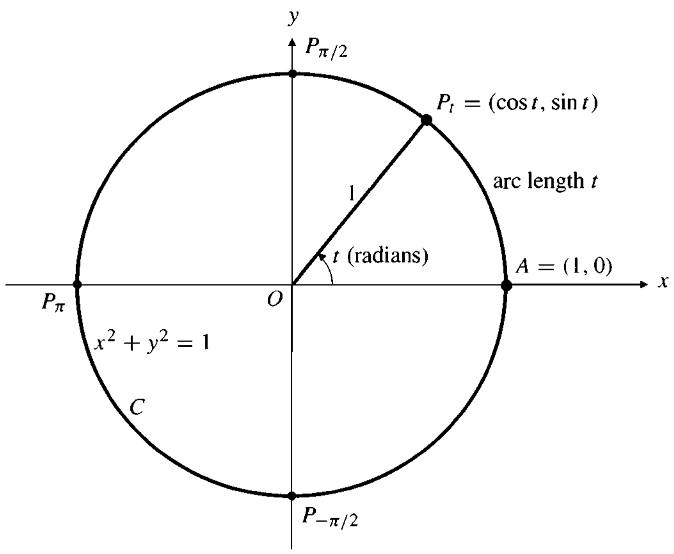



**Figure P.67** If the length of arc $AP_t$ is $t$ units, then angle $AOP_t = t$ radians

**DEFINITION 6**

The **radian measure** of angle $AOP_t$ is $t$ radians:

$$\angle AOP_t = t \text{ radians.}$$

We are more used to measuring angles in **degrees**. Since $P_\pi$ is the point $(-1, 0)$, halfway ($\pi$ units of distance) around $C$ from $A$, we have

$$\pi \text{ radians} = 180°.$$

To convert degrees to radians, multiply by $\pi/180$; to convert radians to degrees, multiply by $180/\pi$.

**Angle convention**

In calculus it is assumed that all angles are measured in radians unless degrees or other units are stated explicitly. When we talk about the angle $\pi/3$, we mean $\pi/3$ radians (which is 60°), not $\pi/3$ degrees.

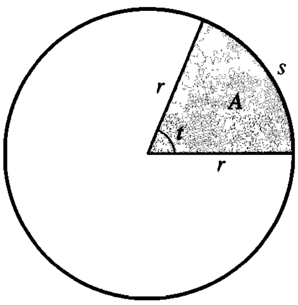



**Figure P.68** Arc length $s = rt$
Sector area $A = r^2t/2$

**EXAMPLE 1** **Arc length and sector area.** An arc of a circle of radius $r$ subtends an angle $t$ at the centre of the circle. Find the length $s$ of the arc and the area $A$ of the sector lying between the arc and the centre of the circle.

***Solution*** The length $s$ of the arc is the same fraction of the circumference $2\pi r$ of the circle that the angle $t$ is of a complete revolution $2\pi$ radians (or 360°). Thus,

$$s = \frac{t}{2\pi}(2\pi r) = rt \text{ units.}$$

Similarly, the area $A$ of the circular sector (Figure P.68) is the same fraction of the area $\pi r^2$ of the whole circle:

$$A = \frac{t}{2\pi}(\pi r^2) = \frac{r^2 t}{2} \text{ units}^2.$$

(We will show that the area of a circle of radius $r$ is $\pi r^2$ in Section 1.1.)

Using the procedure described above, we can find the point $P_t$ corresponding to any real number $t$, positive or negative. We define $\cos t$ and $\sin t$ to be the coordinates of $P_t$. (See Figure P.69.)

**DEFINITION 7**

**Cosine and sine**

For any real $t$, the **cosine** of $t$ (abbreviated $\cos t$) and the **sine** of $t$ (abbreviated $\sin t$) are the $x$- and $y$-coordinates of the point $P_t$.

$$\cos t = \text{the } x\text{-coordinate of } P_t$$
$$\sin t = \text{the } y\text{-coordinate of } P_t$$

Because they are defined this way, cosine and sine are often called the **circular functions**. Note that these definitions agree with the ones given earlier for an acute angle. (See formulas (∗) at the beginning of this section.) The triangle involved is $P_tOQ_t$ in Figure P.69.

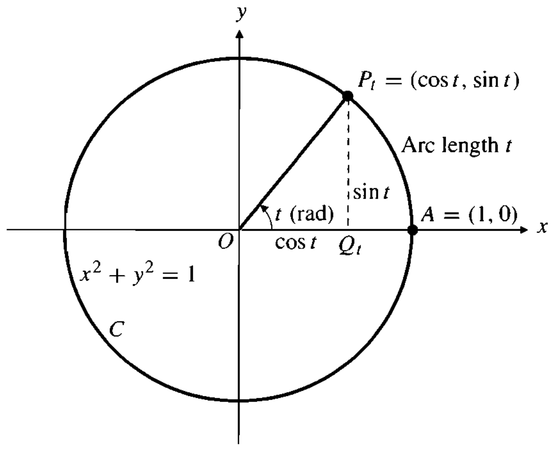



**Figure P.69** The coordinates of $P_t$ are $(\cos t, \sin t)$

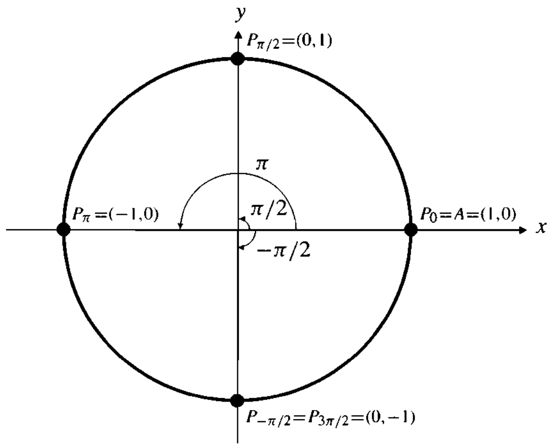



**Figure P.70** Some special angles

**EXAMPLE 2** Examining the coordinates of $P_0 = A$, $P_{\pi/2}$, $P_\pi$, and $P_{-\pi/2} = P_{3\pi/2}$ in Figure P.70, we obtain the following values:

$$\cos 0 = 1 \quad \cos\frac{\pi}{2} = 0 \quad \cos\pi = -1 \quad \cos\left(-\frac{\pi}{2}\right) = \cos\frac{3\pi}{2} = 0$$

$$\sin 0 = 0 \quad \sin\frac{\pi}{2} = 1 \quad \sin\pi = 0 \quad \sin\left(-\frac{\pi}{2}\right) = \sin\frac{3\pi}{2} = -1$$

## Some Useful Identities

Many important properties of $\cos t$ and $\sin t$ follow from the fact that they are coordinates of the point $P_t$ on the circle $C$ with equation $x^2 + y^2 = 1$.

**The range of cosine and sine.** For every real number $t$,

$$-1 \le \cos t \le 1 \qquad \text{and} \qquad -1 \le \sin t \le 1.$$

**The Pythagorean identity.** The coordinates $x = \cos t$ and $y = \sin t$ of $P_t$ must satisfy the equation of the circle. Therefore, for every real number $t$,

$$\cos^2 t + \sin^2 t = 1.$$

(Note that $\cos^2 t$ means $(\cos t)^2$, not $\cos(\cos t)$. This is an unfortunate notation, but it is used everywhere in technical literature, so you have to get used to it!)

**Periodicity.** Since $C$ has circumference $2\pi$, adding $2\pi$ to $t$ causes the point $P_t$ to go one extra complete revolution around $C$ and end up in the same place: $P_{t+2\pi} = P_t$. Thus, for every $t$,

$$\cos(t + 2\pi) = \cos t \quad \text{and} \quad \sin(t + 2\pi) = \sin t.$$

This says that cosine and sine are **periodic** with period $2\pi$.

**Cosine is an even function. Sine is an odd function.** Since the circle $x^2 + y^2 = 1$ is symmetric about the $x$-axis, the points $P_{-t}$ and $P_t$ have the same $x$-coordinates and opposite $y$-coordinates (Figure P.71).

$$\cos(-t) = \cos t \quad \text{and} \quad \sin(-t) = -\sin t.$$

**Complementary angle identities.** Two angles are complementary if their sum is $\pi/2$ (or 90°). The points $P_{(\pi/2)-t}$ and $P_t$ are reflections of each other in the line $y = x$ (Figure P.72), so the $x$-coordinate of one is the $y$-coordinate of the other and vice versa. Thus,

$$\cos\left(\frac{\pi}{2} - t\right) = \sin t \quad \text{and} \quad \sin\left(\frac{\pi}{2} - t\right) = \cos t.$$

**Supplementary angle identities.** Two angles are supplementary if their sum is $\pi$ (or 180°). Since the circle is symmetric about the $y$-axis, $P_{\pi-t}$ and $P_t$ have the same $y$-coordinates and opposite $x$-coordinates. (See Figure P.73.) Thus,

$$\cos(\pi - t) = -\cos t \quad \text{and} \quad \sin(\pi - t) = \sin t.$$

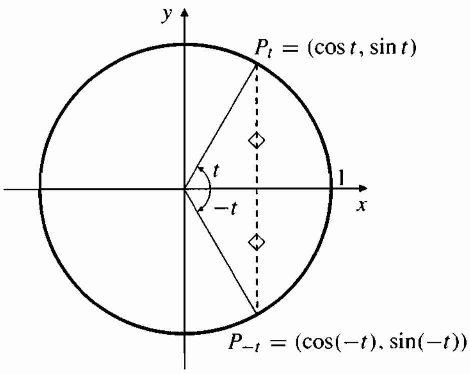



**Figure P.71** $\cos(-t) = \cos t$
$\sin(-t) = -\sin t$

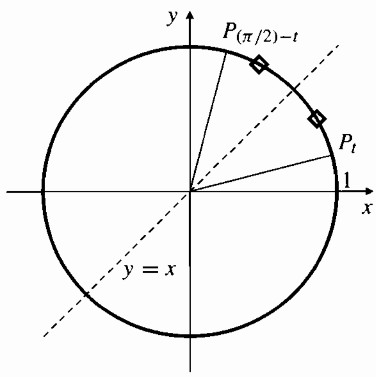



**Figure P.72** $\cos((\pi/2) - t) = \sin t$
$\sin((\pi/2) - t) = \cos t$

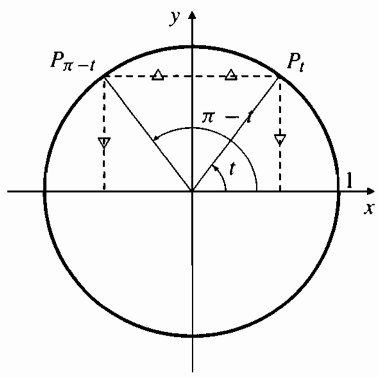



**Figure P.73** $\cos(\pi - t) = -\cos t$
$\sin(\pi - t) = \sin t$

## Some Special Angles

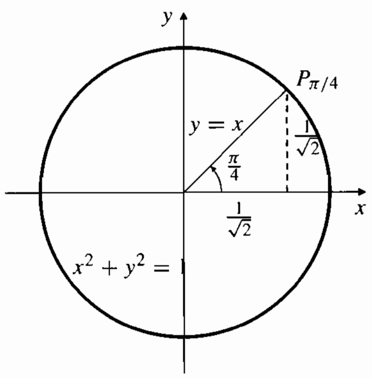



**Figure P.74** $\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

**EXAMPLE 3** Find the sine and cosine of $\pi/4$ (i.e., 45°).

***Solution*** The point $P_{\pi/4}$ lies in the first quadrant on the line $x = y$. To find its coordinates, substitute $y = x$ into the equation $x^2 + y^2 = 1$ of the circle, obtaining $2x^2 = 1$. Thus $x = y = 1/\sqrt{2}$ (see Figure P.74), and

$$\cos(45°) = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}, \qquad \sin(45°) = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}.$$

---

**EXAMPLE 4** Find the values of sine and cosine of the angles $\pi/3$ (or 60°) and $\pi/6$ (or 30°).

**Solution** The point $P_{\pi/3}$ and the points $O(0, 0)$ and $A(1, 0)$ are the vertices of an equilateral triangle with edge length 1 (see Figure P.75). Thus $P_{\pi/3}$ has $x$-coordinate 1/2 and $y$-coordinate $\sqrt{1-(1/2)^2} = \sqrt{3}/2$, and

$$\cos(60^\circ) = \cos\frac{\pi}{3} = \frac{1}{2}, \qquad \sin(60^\circ) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}.$$

Since $\dfrac{\pi}{6} = \dfrac{\pi}{2} - \dfrac{\pi}{3}$, the complementary angle identities now tell us that

$$\cos(30^\circ) = \cos\frac{\pi}{6} = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}, \qquad \sin(30^\circ) = \sin\frac{\pi}{6} = \cos\frac{\pi}{3} = \frac{1}{2}.$$

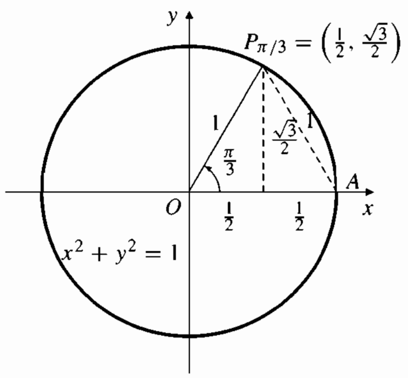



**Figure P.75** $\cos \pi/3 = 1/2$
$\sin \pi/3 = \sqrt{3}/2$

Table 5 summarizes the values of cosine and sine at multiples of 30° and 45° between 0° and 180°. The values for 120°, 135°, and 150° were determined by using the supplementary angle identities; for example,

$$\cos(120^\circ) = \cos\left(\frac{2\pi}{3}\right) = \cos\left(\pi - \frac{\pi}{3}\right) = -\cos\left(\frac{\pi}{3}\right) = -\cos(60^\circ) = -\frac{1}{2}.$$

**Table 5.** Cosines and sines of special angles

| Degrees | 0° | 30° | 45° | 60° | 90° | 120° | 135° | 150° | 180° |
|---|---|---|---|---|---|---|---|---|---|
| Radians | 0 | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ | $\frac{2\pi}{3}$ | $\frac{3\pi}{4}$ | $\frac{5\pi}{6}$ | $\pi$ |
| Cosine | 1 | $\frac{\sqrt{3}}{2}$ | $\frac{1}{\sqrt{2}}$ | $\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | $-\frac{1}{\sqrt{2}}$ | $-\frac{\sqrt{3}}{2}$ | $-1$ |
| Sine | 0 | $\frac{1}{2}$ | $\frac{1}{\sqrt{2}}$ | $\frac{\sqrt{3}}{2}$ | 1 | $\frac{\sqrt{3}}{2}$ | $\frac{1}{\sqrt{2}}$ | $\frac{1}{2}$ | 0 |

**EXAMPLE 5** Find: (a) $\sin(3\pi/4)$ and (b) $\cos(4\pi/3)$.

**Solution** We can draw appropriate triangles in the quadrants where the angles lie to determine the required values. See Figure P.76.

(a) $\sin(3\pi/4) = \sin(\pi - (\pi/4)) = 1/\sqrt{2}$.

(b) $\cos(4\pi/3) = \cos(\pi + (\pi/3)) = -\dfrac{1}{2}$.

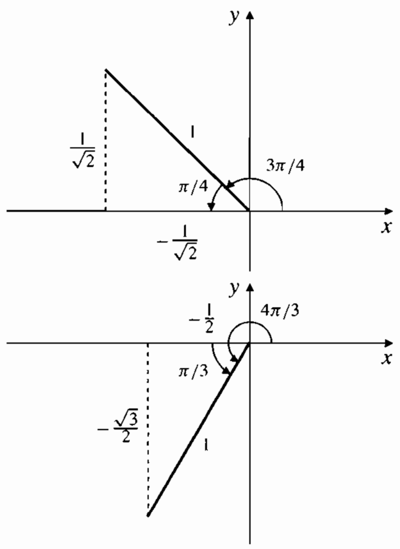



**Figure P.76** Using suitably placed triangles to find trigonometric functions of special angles

While decimal approximations to the values of sine and cosine can be found using a scientific calculator or mathematical tables, it is useful to remember the exact values in the table for angles 0, $\pi/6$, $\pi/4$, $\pi/3$, and $\pi/2$. They occur frequently in applications.

When we treat sine and cosine as functions, we can call the variable they depend on anything we want (e.g., $x$, as we do with other functions), rather than $t$. The graphs of $\cos x$ and $\sin x$ are shown in Figures P.77 and P.78. In both graphs the pattern between $x = 0$ and $x = 2\pi$ repeats over and over to the left and right. Observe that the graph of $\sin x$ is the graph of $\cos x$ shifted to the right a distance $\pi/2$.

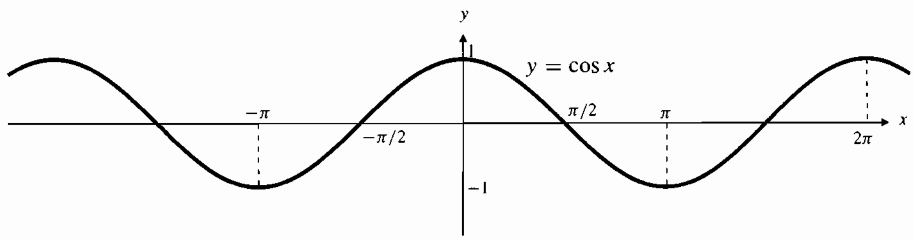



**Figure P.77** The graph of $\cos x$

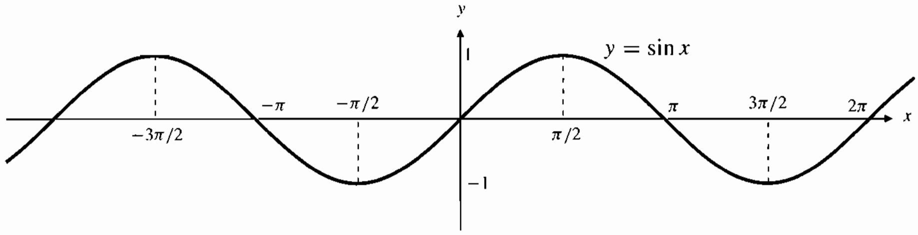



Figure P.78 The graph of $\sin x$

**Remember this!**

When using a scientific calculator to calculate any trigonometric functions, be sure you have selected the proper angular mode: degrees or radians.

## The Addition Formulas

The following formulas enable us to determine the cosine and sine of a sum or difference of two angles in terms of the cosines and sines of those angles.

**THEOREM 2** **Addition Formulas for Cosine and Sine**

$$\cos(s+t) = \cos s\cos t - \sin s\sin t$$
$$\sin(s+t) = \sin s\cos t + \cos s\sin t$$
$$\cos(s-t) = \cos s\cos t + \sin s\sin t$$
$$\sin(s-t) = \sin s\cos t - \cos s\sin t$$

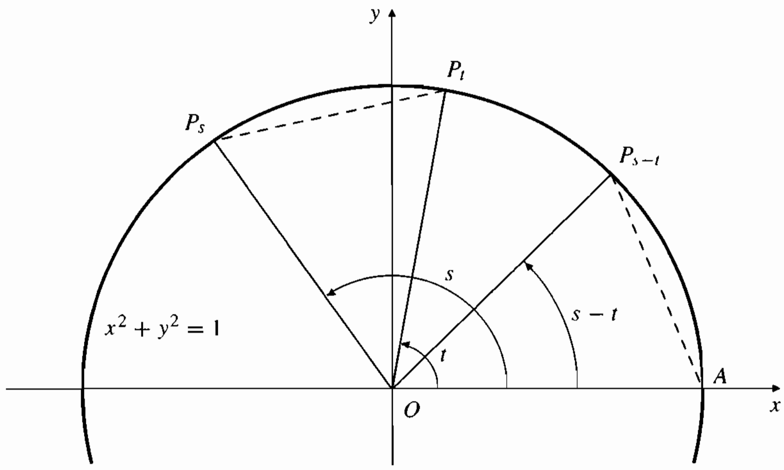



Figure P.79 $P_sP_t = P_{s-t}A$

***PROOF*** We prove the third of these formulas as follows: Let $s$ and $t$ be real numbers and consider the points

$$P_t = (\cos t, \sin t) \qquad P_{s-t} = (\cos(s-t), \sin(s-t))$$
$$P_s = (\cos s, \sin s) \qquad A = (1, 0),$$

as shown in Figure P.79.

The angle $P_tOP_s = s - t$ radians = angle $AOP_{s-t}$, so the distance $P_sP_t$ is equal to the distance $P_{s-t}A$. Therefore, $(P_sP_t)^2 = (P_{s-t}A)^2$. We express these squared distances in terms of coordinates and expand the resulting squares of binomials:

$$(\cos s - \cos t)^2 + (\sin s - \sin t)^2 = (\cos(s-t) - 1)^2 + \sin^2(s-t),$$

$$\cos^2 s - 2\cos s\cos t + \cos^2 t + \sin^2 s - 2\sin s\sin t + \sin^2 t$$
$$= \cos^2(s-t) - 2\cos(s-t) + 1 + \sin^2(s-t).$$

Since $\cos^2 x + \sin^2 x = 1$ for every $x$, this reduces to

$$\cos(s - t) = \cos s \cos t + \sin s \sin t.$$

Replacing $t$ with $-t$ in the formula above, and recalling that $\cos(-t) = \cos t$ and $\sin(-t) = -\sin t$, we have

$$\cos(s + t) = \cos s \cos t - \sin s \sin t.$$

The complementary angle formulas can be used to obtain either of the addition formulas for sine:

$$\begin{aligned}\sin(s + t) &= \cos\left(\frac{\pi}{2} - (s + t)\right)\\ &= \cos\left(\left(\frac{\pi}{2} - s\right) - t\right)\\ &= \cos\left(\frac{\pi}{2} - s\right)\cos t + \sin\left(\frac{\pi}{2} - s\right)\sin t\\ &= \sin s \cos t + \cos s \sin t,\end{aligned}$$

and the other formula again follows if we replace $t$ with $-t$.

**EXAMPLE 6** Find the value of $\cos(\pi/12) = \cos 15°$.

***Solution***

$$\begin{aligned}\cos\frac{\pi}{12} &= \cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \cos\frac{\pi}{3}\cos\frac{\pi}{4} + \sin\frac{\pi}{3}\sin\frac{\pi}{4}\\ &= \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) = \frac{1 + \sqrt{3}}{2\sqrt{2}}\end{aligned}$$

From the addition formulas, we obtain as special cases certain useful formulas called **double-angle formulas**. Put $s = t$ in the addition formulas for $\sin(s + t)$ and $\cos(s + t)$ to get

$$\begin{aligned}\sin 2t &= 2\sin t \cos t \qquad \text{and}\\ \cos 2t &= \cos^2 t - \sin^2 t\\ &= 2\cos^2 t - 1 \qquad (\text{using } \sin^2 t + \cos^2 t = 1)\\ &= 1 - 2\sin^2 t\end{aligned}$$

Solving the last two formulas for $\cos^2 t$ and $\sin^2 t$, we obtain

$$\cos^2 t = \frac{1 + \cos 2t}{2} \qquad \text{and} \qquad \sin^2 t = \frac{1 - \cos 2t}{2},$$

which are sometimes called **half-angle formulas** because they are used to express trigonometric functions of half of the angle $2t$. Later we will find these formulas useful when we have to integrate powers of $\cos x$ and $\sin x$.

## Other Trigonometric Functions

There are four other trigonometric functions—tangent (tan), cotangent (cot), secant (sec), and cosecant (csc)—each defined in terms of cosine and sine. Their graphs are shown in Figures P.80–P.83.

**DEFINITION 8**

**Tangent, cotangent, secant, and cosecant**

$$\tan t = \frac{\sin t}{\cos t} \qquad \sec t = \frac{1}{\cos t}$$
$$\cot t = \frac{\cos t}{\sin t} = \frac{1}{\tan t} \qquad \csc t = \frac{1}{\sin t}$$

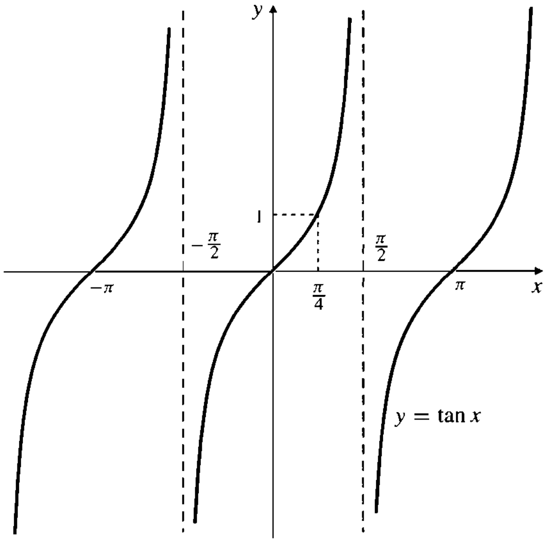



**Figure P.80** The graph of $\tan x$

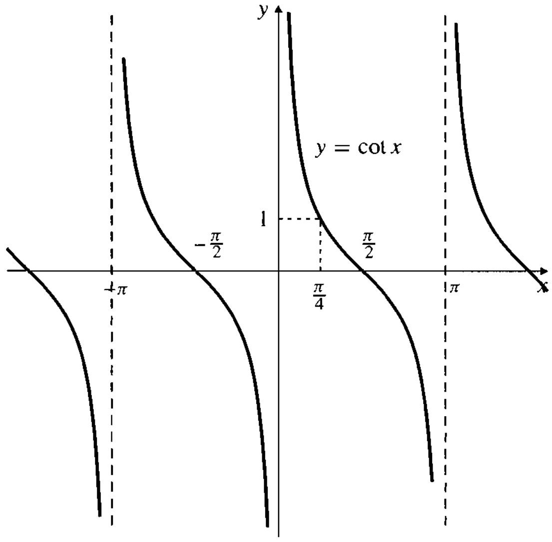



**Figure P.81** The graph of $\cot x$

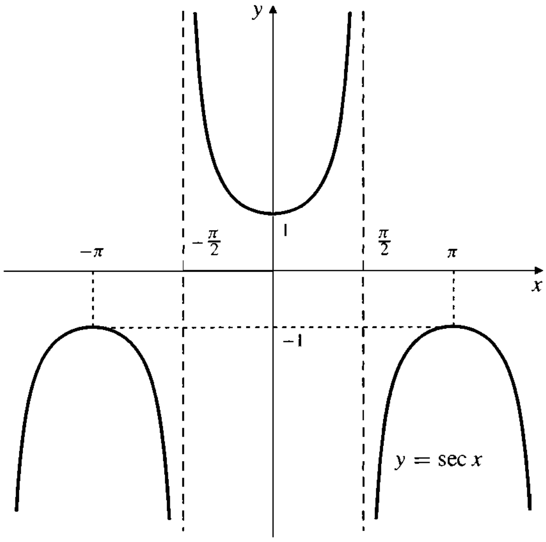



**Figure P.82** The graph of $\sec x$

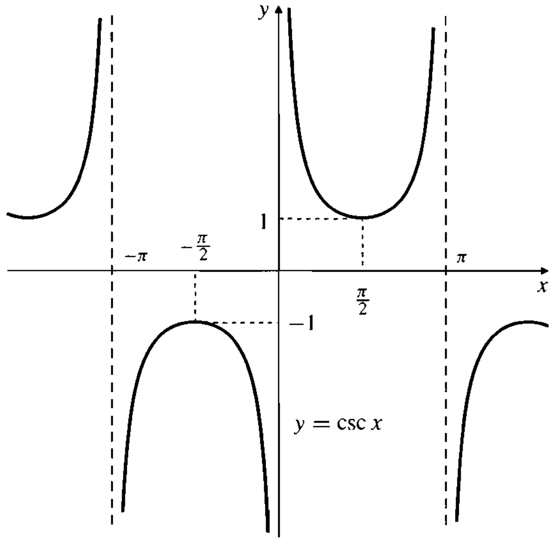



**Figure P.83** The graph of $\csc x$

Observe that each of these functions is undefined (and its graph approaches vertical asymptotes) at points where the function in the denominator of its defining fraction has value 0. Observe also that tangent, cotangent, and cosecant are odd functions and secant is an even function. Since $|\sin x| \le 1$ and $|\cos x| \le 1$ for all $x$, $|\csc x| \ge 1$ and $|\sec x| \ge 1$ for all $x$ where they are defined.

The three functions sine, cosine, and tangent are called the **primary trigonometric functions**, while their reciprocals cosecant, secant, and cotangent are called the **secondary trigonometric functions**. Scientific calculators usually just implement the primary functions; you can use the reciprocal key to find values of the corresponding secondary functions. Figure P.84 shows a useful pattern called the "CAST rule" to help you remember where the primary functions are positive. All three are positive in the first quadrant, marked A. Of the three, only sine is positive in the second quadrant S, only tangent in the third quadrant T, and only cosine in the fourth quadrant C.

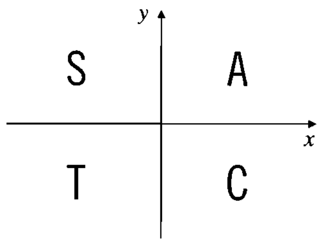



**Figure P.84** The CAST rule

**EXAMPLE 7** Find the sine and tangent of the angle $\theta$ in $\left[\pi, \frac{3\pi}{2}\right]$ for which we have $\cos\theta = -\frac{1}{3}$.

***Solution*** From the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$ we get

$$\sin^2\theta = 1 - \frac{1}{9} = \frac{8}{9}, \qquad \text{so} \quad \sin\theta = \pm\sqrt{\frac{8}{9}} = \pm\frac{2\sqrt{2}}{3}.$$

The requirement that $\theta$ should lie in $[\pi, 3\pi/2]$ makes $\theta$ a third quadrant angle. Its sine is therefore negative. We have

$$\sin\theta = -\frac{2\sqrt{2}}{3} \qquad \text{and} \qquad \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-2\sqrt{2}/3}{-1/3} = 2\sqrt{2}.$$

Like their reciprocals cosine and sine, the functions secant and cosecant are periodic with period $2\pi$. Tangent and cotangent, however, have period $\pi$ because

$$\tan(x+\pi) = \frac{\sin(x+\pi)}{\cos(x+\pi)} = \frac{\sin x \cos\pi + \cos x \sin\pi}{\cos x \cos\pi - \sin x \sin\pi} = \frac{-\sin x}{-\cos x} = \tan x.$$

Dividing the Pythagorean identity $\sin^2 x + \cos^2 x = 1$ by $\cos^2 x$ and $\sin^2 x$, respectively, leads to two useful alternative versions of that identity:

$$1 + \tan^2 x = \sec^2 x \qquad \text{and} \qquad 1 + \cot^2 x = \csc^2 x.$$

Addition formulas for tangent and cotangent can be obtained from those for sine and cosine. For example,

$$\tan(s+t) = \frac{\sin(s+t)}{\cos(s+t)} = \frac{\sin s \cos t + \cos s \sin t}{\cos s \cos t - \sin s \sin t}.$$

Now divide the numerator and denominator of the fraction on the right by $\cos s \cos t$ to get

$$\tan(s+t) = \frac{\tan s + \tan t}{1 - \tan s \tan t}.$$

Replacing $t$ by $-t$ leads to

$$\tan(s-t) = \frac{\tan s - \tan t}{1 + \tan s \tan t}.$$

## Maple Calculations

Maple knows all six trigonometric functions and can calculate their values and manipulate them in other ways. It assumes the arguments of the trigonometric functions are in radians.

```
> evalf(sin(30)); evalf(sin(Pi/6));
```

$$-.9880316241$$

$$.5000000000$$

Note that the constant Pi (with an uppercase P) is known to Maple. The `evalf()` function converts its argument to a number expressed as a floating point decimal with 10 significant digits. (This precision can be changed by defining a new value for the variable `Digits`.) Without it, the sine of 30 radians would have been left unexpanded because it is not an integer.

```
> Digits := 20; evalf(100*Pi); sin(30);
```

$$Digits := 20$$

$$314.15926535897932385$$

$$sin(30)$$

It is often useful to expand trigonometric functions of multiple angles to powers of sine and cosine, and vice versa.

```
> expand(sin(5*x));
```

$$16\sin(x)\cos(x)^4 - 12\sin(x)\cos(x)^2 + \sin(x)$$

```
> combine((cos(x))^5, trig);
```

$$\frac{1}{16}\cos(5x) + \frac{5}{16}\cos(3x) + \frac{5}{8}\cos(x)$$

Other trigonometric functions can be converted to expressions involving sine and cosine.

```
> convert(tan(4*x)*(sec(4*x))^2, sincos); combine(%,trig)
```

$$\frac{\sin(4x)}{\cos(4x)^3}$$

$$4\frac{\sin(4x)}{\cos(12x) + 3\cos(4x)}$$

The % in the last command refers to the result of the previous calculation.

## Trigonometry Review

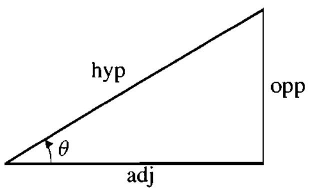



**Figure P.85**

The trigonometric functions are so called because they are often used to express the relationships between the sides and angles of a triangle. As we observed at the beginning of this section, if $\theta$ is one of the acute angles in a right-angled triangle, we can refer to the three sides of the triangle as adj (side adjacent $\theta$), opp (side opposite $\theta$), and hyp (hypotenuse). (SeeFigure P.85.) The trigonometric functions of $\theta$ can then be expressed as ratios of these sides, in particular:

$$\sin\theta = \frac{\text{opp}}{\text{hyp}}, \qquad \cos\theta = \frac{\text{adj}}{\text{hyp}}, \qquad \tan\theta = \frac{\text{opp}}{\text{adj}}.$$

**EXAMPLE 8** Find the unknown sides $x$ and $y$ of the triangle in Figure P.86.

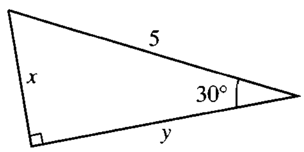



**Figure P.86**

***Solution*** Here, $x$ is the side opposite and $y$ is the side adjacent the 30° angle. The hypotenuse of the triangle is 5 units. Thus,

$$\frac{x}{5} = \sin 30° = \frac{1}{2} \quad \text{and} \quad \frac{y}{5} = \cos 30° = \frac{\sqrt{3}}{2},$$

so $x = \dfrac{5}{2}$ units and $y = \dfrac{5\sqrt{3}}{2}$ units.

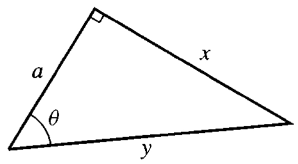



**Figure P.87**

**EXAMPLE 9** For the triangle in Figure P.87, express sides $x$ and $y$ in terms of side $a$ and angle $\theta$.

***Solution*** The side $x$ is opposite the angle $\theta$, and $y$ is the hypotenuse. The side adjacent $\theta$ is $a$. Thus,

$$\frac{x}{a} = \tan\theta \quad \text{and} \quad \frac{a}{y} = \cos\theta.$$

Hence, $x = a\,\tan\theta$ and $y = \dfrac{a}{\cos\theta} = a\,\sec\theta$.

When dealing with general (not necessarily right-angled) triangles, it is often convenient to label the vertices with capital letters, which also denote the angles at those vertices, and refer to the sides opposite those vertices by the corresponding lowercase letters. See Figure P.88. Relationships between the sides $a$, $b$, and $c$ and opposite angles $A$, $B$, and $C$ of an arbitrary triangle $ABC$ are given by the following formulas, called the **Sine Law** and the **Cosine Law**.

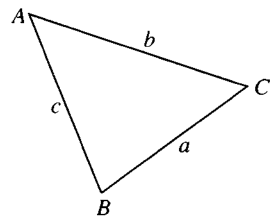



**Figure P.88** In this triangle the sides are named to correspond to the opposite angles

## THEOREM 3

**Sine Law:** $$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

**Cosine Law:**
$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$

***PROOF*** See Figure P.89. Let $h$ be the length of the perpendicular from $A$ to the side $BC$. From right-angled triangles (and using $\sin(\pi - t) = \sin t$ if required), we get $c\sin B = h = b\sin C$. Thus $(\sin B)/b = (\sin C)/c$. By the symmetry of the formulas (or by dropping a perpendicular to another side), both fractions must be equal to $(\sin A)/a$, so the Sine Law is proved. For the Cosine Law, observe that

$$\begin{aligned} c^2 &= \begin{cases} h^2 + (a - b\cos C)^2 & \text{if } C \le \dfrac{\pi}{2} \\[2ex] h^2 + (a + b\cos(\pi - C))^2 & \text{if } C > \dfrac{\pi}{2} \end{cases} \\ &= h^2 + (a - b\cos C)^2 \qquad (\text{since } \cos(\pi - C) = -\cos C) \\ &= b^2\sin^2 C + a^2 - 2ab\cos C + b^2\cos^2 C \\ &= a^2 + b^2 - 2ab\cos C. \end{aligned}$$

The other versions of the Cosine Law can be proved in a similar way.

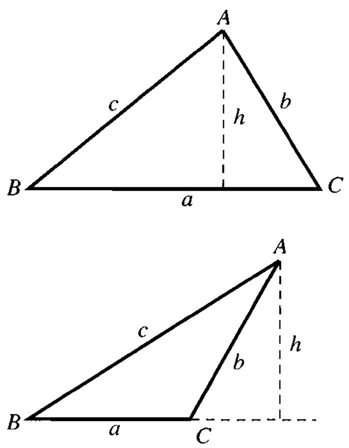



**Figure P.89**

**EXAMPLE 10** A triangle has sides $a = 2$ and $b = 3$ and angle $C = 40°$. Find side $c$ and the sine of angle $B$.

***Solution*** From the third version of the Cosine Law:

$$c^2 = a^2 + b^2 - 2ab\cos C = 4 + 9 - 12\cos 40° \approx 13 - 12 \times 0.766 = 3.808.$$

Side $c$ is about $\sqrt{3.808} = 1.951$ units in length. Now using Sine Law we get

$$\sin B = b\,\frac{\sin C}{c} \approx 3 \times \frac{\sin 40°}{1.951} \approx \frac{3 \times 0.6428}{1.951} \approx 0.988.$$

A triangle is uniquely determined by any one of the following sets of data (which correspond to the known cases of congruency of triangles in classical geometry):

1. two sides and the angle contained between them (e.g., Example 10);
2. three sides, no one of which exceeds the sum of the other two in length;
3. two angles and one side; or
4. the hypotenuse and one other side of a right-angled triangle.

In such cases you can always find the unknown sides and angles by using the Pythagorean Theorem or the Sine and Cosine Laws, and the fact that the sum of the three angles of a triangle is 180° (or $\pi$ radians).

A triangle is not determined uniquely by two sides and a noncontained angle; there may exist no triangle, one right-angled triangle, or two triangles having such data.

**EXAMPLE 11** In triangle $ABC$, angle $B = 30°$, $b = 2$, and $c = 3$. Find $a$.

***Solution*** This is one of the ambiguous cases. By the Cosine Law,

$$\begin{aligned} b^2 &= a^2 + c^2 - 2ac\cos B \\ 4 &= a^2 + 9 - 6a(\sqrt{3}/2). \end{aligned}$$

Therefore, $a$ must satisfy the equation $a^2 - 3\sqrt{3}a + 5 = 0$. Solving this equation using the quadratic formula, we obtain

$$\begin{aligned} a &= \frac{3\sqrt{3} \pm \sqrt{27 - 20}}{2} \\ &\approx 1.275 \quad \text{or} \quad 3.921 \end{aligned}$$

There are two triangles with the given data, as shown in Figure P.90.

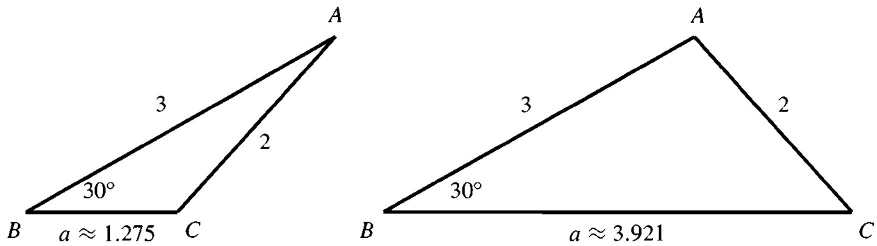



**Figure P.90** Two triangles with $b = 2$, $c = 3$, $B = 30°$

## EXERCISES P.7

Find the values of the quantities in Exercises 1–6 using various formulas presented in this section. Do not use tables or a calculator.

**1.** $\cos\dfrac{3\pi}{4}$ **2.** $\tan -\dfrac{3\pi}{4}$ **3.** $\sin\dfrac{2\pi}{3}$

**4.** $\sin\dfrac{7\pi}{12}$ **5.** $\cos\dfrac{5\pi}{12}$ **6.** $\sin\dfrac{11\pi}{12}$

In Exercises 7–12, express the given quantity in terms of $\sin x$ and $\cos x$.

**7.** $\cos(\pi + x)$ **8.** $\sin(2\pi - x)$ **9.** $\sin\left(\dfrac{3\pi}{2} - x\right)$

**10.** $\cos\left(\dfrac{3\pi}{2} + x\right)$ **11.** $\tan x + \cot x$ **12.** $\dfrac{\tan x - \cot x}{\tan x + \cot x}$

In Exercises 13–16, prove the given identities.

**13.** $\cos^4 x - \sin^4 x = \cos(2x)$

**14.** $\dfrac{1 - \cos x}{\sin x} = \dfrac{\sin x}{1 + \cos x} = \tan\dfrac{x}{2}$

**15.** $\dfrac{1 - \cos x}{1 + \cos x} = \tan^2\dfrac{x}{2}$

**16.** $\dfrac{\cos x - \sin x}{\cos x + \sin x} = \sec 2x - \tan 2x$

**17.** Express $\sin 3x$ in terms of $\sin x$ and $\cos x$.

**18.** Express $\cos 3x$ in terms of $\sin x$ and $\cos x$.

In Exercises 19–22, sketch the graph of the given function. What is the period of the function?

**19.** $f(x) = \cos 2x$

**20.** $f(x) = \sin \dfrac{x}{2}$

**21.** $f(x) = \sin \pi x$

**22.** $f(x) = \cos \dfrac{\pi x}{2}$

**23.** Sketch the graph of $y = 2\cos\left(x - \dfrac{\pi}{3}\right)$.

**24.** Sketch the graph of $y = 1 + \sin\left(x + \dfrac{\pi}{4}\right)$.

In Exercises 25–30, one of $\sin\theta$, $\cos\theta$, and $\tan\theta$ is given. Find the other two if $\theta$ lies in the specified interval.

**25.** $\sin\theta = \dfrac{3}{5}$, $\quad\theta$ in $\left[\dfrac{\pi}{2}, \pi\right]$

**26.** $\tan\theta = 2$, $\quad\theta$ in $\left[0, \dfrac{\pi}{2}\right]$

**27.** $\cos\theta = \dfrac{1}{3}$, $\quad\theta$ in $\left[-\dfrac{\pi}{2}, 0\right]$

**28.** $\cos\theta = -\dfrac{5}{13}$, $\quad\theta$ in $\left[\dfrac{\pi}{2}, \pi\right]$

**29.** $\sin\theta = \dfrac{-1}{2}$, $\quad\theta$ in $\left[\pi, \dfrac{3\pi}{2}\right]$

**30.** $\tan\theta = \dfrac{1}{2}$, $\quad\theta$ in $\left[\pi, \dfrac{3\pi}{2}\right]$

### Trigonometry Review

In Exercises 31–42, $ABC$ is a triangle with a right angle at $C$. The sides opposite angles $A$, $B$, and $C$ are $a$, $b$, and $c$, respectively. (See Figure P.91.)

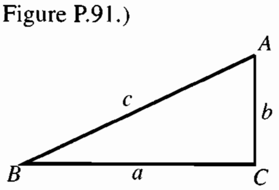



Figure P.91

**31.** Find $a$ and $b$ if $c = 2$, $B = \dfrac{\pi}{3}$.

**32.** Find $a$ and $c$ if $b = 2$, $B = \dfrac{\pi}{3}$.

**33.** Find $b$ and $c$ if $a = 5$, $B = \dfrac{\pi}{6}$.

**34.** Express $a$ in terms of $A$ and $c$.

**35.** Express $a$ in terms of $A$ and $b$.

**36.** Express $a$ in terms of $B$ and $c$.

**37.** Express $a$ in terms of $B$ and $b$.

**38.** Express $c$ in terms of $A$ and $a$.

**39.** Express $c$ in terms of $A$ and $b$.

**40.** Express $\sin A$ in terms of $a$ and $c$.

**41.** Express $\sin A$ in terms of $b$ and $c$.

**42.** Express $\sin A$ in terms of $a$ and $b$.

In Exercises 43–50, $ABC$ is an arbitrary triangle with sides $a$, $b$, and $c$, opposite to angles $A$, $B$, and $C$, respectively. (See Figure P.92.) Find the indicated quantities. Use tables or a scientific calculator if necessary.

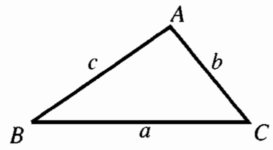



Figure P.92

**43.** Find $\sin B$ if $a = 4$, $b = 3$, $A = \dfrac{\pi}{4}$.

**44.** Find $\cos A$ if $a = 2$, $b = 2$, $c = 3$.

**45.** Find $\sin B$ if $a = 2$, $b = 3$, $c = 4$.

**46.** Find $c$ if $a = 2$, $b = 3$, $C = \dfrac{\pi}{4}$.

**47.** Find $a$ if $c = 3$, $A = \dfrac{\pi}{4}$, $B = \dfrac{\pi}{3}$.

**48.** Find $c$ if $a = 2$, $b = 3$, $C = 35°$.

**49.** Find $b$ if $a = 4$, $B = 40°$, $C = 70°$.

**50.** Find $c$ if $a = 1$, $b = \sqrt{2}$, $A = 30°$. (There are two possible answers.)

**51.** Two guy wires stretch from the top $T$ of a vertical pole to points $B$ and $C$ on the ground, where $C$ is 10 m closer to the base of the pole than is $B$. If wire $BT$ makes an angle of 35° with the horizontal, and wire $CT$ makes an angle of 50° with the horizontal, how high is the pole?

**52.** Observers at positions $A$ and $B$ 2 km apart simultaneously measure the angle of elevation of a weather balloon to be 40° and 70°, respectively. If the balloon is directly above a point on the line segment between $A$ and $B$, find the height of the balloon.

**53.** Show that the area of triangle $ABC$ is given by $(1/2)ab\sin C = (1/2)bc\sin A = (1/2)ca\sin B$.

**54.** Show that the area of triangle $ABC$ is given by $\sqrt{s(s-a)(s-b)(s-c)}$, where $s = (a+b+c)/2$ is the semi-perimeter of the triangle.

This symbol is used throughout the book to indicate an exercise that is somewhat more difficult and/or theoretical than most exercises.